Combinatorial Group Theory

Lecture notes summer semester 2022

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$$M_{11} = \langle x, y, z \mid x^{11} = y^5 = z^4 = (xz)^3 = 1,\ yxy^{-1} = x^4,\ zyz^{-1} = y^2 \rangle$$

x	x^{-1}	y	y^{-1}	z	z^{-1}
2	3	1	1	4	4
5	1	6	7	8	9
1	10	11	12	11	13
14	15	4	4	1	1
6	2	13	11	5	5
7	5	10	2	16	16
17	6	2	17	14	18
19	9	9	18	10	2
8	16	14	8	2	10
3	11	17	6	9	8
10	12	5	3	12	3
11	13	3	13	13	11
12	17	12	5	3	12
20	4	16	9	17	7
4	18	21	22	21	19
9	20	18	14	6	6
13	7	7	10	18	14
15	21	8	16	7	17
22	8	22	20	15	22
16	14	19	21	20	20
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Warning: This is an AI-translated version of my German lecture notes, performed by *Gemini 3 Flash Preview*. I have not checked whether Gemini introduced errors. Use with care!

Preface

In combinatorial group theory, groups are mainly studied by means of generators and relations. Naturally, one must therefore deal for the most part with infinite objects. One of the most important motivations is the *Burnside problem*: Is every finitely generated group with finite exponent finite? Since geometric arguments are occasionally used (for example with Coxeter groups), one alternatively speaks of *geometric group theory*.

These lecture notes were created as part of a $3 + 1$ lecture in the summer semester 2022 at Leibniz Universität Hannover. The lecture follows my group theory lecture from the winter semester 2020/21 and presupposes corresponding knowledge (references are marked with GT). In some places, there are duplications in both lectures. The reasons for this are:

- To recall knowledge to mind.
- To present alternative proof methods.
- Topics that are indeed in the group theory notes, but were not covered there due to time constraints (example: Schur extensions).

Chapters 3, 5, 8 and 11–15 were not covered (therefore more errors are to be expected therein). In order to make calculating with (realistic) examples practicable, we provide commands for the free computer algebra system GAP in many places. At this point, many thanks to Thomas Breuer for useful advice. I also thank Annika Bartelt, Luca Blaas, Adrian Homma, Scheima Obeidi, Claude Sonnet (4.6) and Tim Wittenberg for several error reports.

Literature:

- M. Hall, *The Theory of Groups*, 4th printing, The Macmillan Company, New York, 1963[1]
- D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd edition, Springer, New York, 1996[2]
- W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory*, 2nd edition, Dover, Mineola, 2004[3]
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- J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1994
- D. L. Johnson, *Presentations of Groups*, 2nd edition, Cambridge University Press, Cambridge, 1997
- H. S. M. Coxeter, W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th edition, Springer, Berlin, 1980
- The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.1*; 2021, (`https://www.gap-system.org`).

[1] Treats finite and infinite groups equally. Easy to read despite its age.
[2] Somewhat more modern and extensive than Hall.
[3] Unchanged reprint of the 1975 edition. Small font, much continuous text, difficult to read.

1 Free Groups

Remark 1.1. A basis B of a vector space V can be characterized by the following property: For every vector space W, every map $B \to W$ has exactly one linear extension $V \to W$. In this way, one can define bases for arbitrary (non-abelian) groups. In contrast to vector spaces, however, most groups do not possess bases.

Definition 1.2 (Universal Property)**.** A group F is called *free* with respect to a subset $X \subseteq F$ if for every group G and every map $\sigma\colon X \to G$ there exists exactly one homomorphism $\widehat{\sigma}\colon F \to G$ with $\widehat{\sigma}(x) = \sigma(x)$ for all $x \in X$ (i. e. $\widehat{\sigma}$ is an extension of σ).

Example 1.3.

(i) The trivial group $F = 1$ is the only free group with respect to $X = \varnothing$, because every non-trivial group F possesses at least two endomorphisms.

(ii) The group $\mathbb{Z}$ is free with respect to $X = \{1\}$, because every map $\sigma\colon X \to G$ possesses the unique extension $\mathbb{Z} \to G$, $n \mapsto \sigma(1)^n$.

Definition 1.4. Let X be a set, which we call an *alphabet*. The elements of X are called *letters*. Let W be the set of all *words* of the form $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ with $n \in \mathbb{N}_0$, $x_1, \dots, x_n \in X$ and $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$. Here $|w| := n$ is called the *length* of w. For $n = 0$ one obtains the *empty* word $w = 1$. If $x_i \neq x_{i+1}$ or $\epsilon_i = \epsilon_{i+1}$ for $i = 1, \dots, n-1$, then w is called *reduced*. Obviously, one can transform every word w into a reduced word by successively deleting parts of the form xx^{-1} or $x^{-1}x$. Two words $v, w \in W$ are called *equivalent* if they can be transformed into the same reduced word. This is an equivalence relation on W. The set of equivalence classes $F_X := \{[w] : w \in W\}$ then forms a group with respect to concatenation, i. e.

$$[w][v] := [wv] \qquad [w], [v] \in F_X.$$

The neutral element is the equivalence class of the empty word $[1]$. The inverse of $[x_1^{\epsilon_1} \dots x_n^{\epsilon_n}]$ is $[x_n^{-\epsilon_n} \dots x_1^{-\epsilon_1}]$. By identifying $x \in X$ with $[x] \in F_X$, one can assume $X \subseteq F_X$.

Theorem 1.5.

(i) F_X is free with respect to X.

(ii) Every free group with respect to X is isomorphic to F_X.

(iii) It holds that $F_X \cong F_Y$ if and only if X and Y have the same cardinality.

Proof.

(i) Let G be a group and $\sigma\colon X \to G$. For $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n} \in W$ we define

$$\widehat{\sigma}(w) := \sigma(x_1)^{\epsilon_1} \dots \sigma(x_n)^{\epsilon_n} \in G.$$

For equivalent words $v, w \in W$, it clearly holds that $\widehat{\sigma}(v) = \widehat{\sigma}(w)$. Thus $\widehat{\sigma}$ induces a well-defined map $F_X \to G$, which we also denote by $\widehat{\sigma}$. Because of $\widehat{\sigma}(wv) = \widehat{\sigma}(w)\widehat{\sigma}(v)$ for $w, v \in W$, $\widehat{\sigma}$ is a homomorphism. Because of $F_X = \langle X \rangle$, $\widehat{\sigma}$ is uniquely determined by σ.

(iii) Let $\sigma\colon X \to Y$ be a bijection. Then there exist homomorphisms $\alpha\colon F_X \to F_Y$ and $\beta\colon F_Y \to F_X$ with $\alpha_{|X} = \sigma$ and $\beta_{|Y} = \sigma^{-1}$. Thus $\alpha\beta\colon F_Y \to F_Y$ is an extension of id_Y. From the universal property it follows that $\alpha\beta = \mathrm{id}_{F_Y}$. Analogously one shows $\beta\alpha = \mathrm{id}_{F_X}$. Thus α is an isomorphism between F_X and F_Y.

Conversely, let an isomorphism $\alpha\colon F_X \to F_Y$ be given. We consider

$$N_X := \langle g^2, [g,h] : g,h \in F_X \rangle \trianglelefteq F_X$$

and $\overline{F_X} := F_X/N_X$. Because of $\alpha(N_X) = \langle \alpha(g), [\alpha(g),\alpha(h)] : g,h \in F_X\rangle = N_Y$, it holds that $\overline{F_X} \cong \overline{F_Y}$. By construction, $\overline{F_X}$ is an abelian group with $\overline{g}^2 = 1$ for all $\overline{g} \in \overline{F_X}$. Through the scalar multiplication $\lambda\overline{g} := \overline{g}^\lambda$ for $\lambda \in \mathbb{F}_2$, $\overline{F_X}$ becomes an $\mathbb{F}_2$-vector space (an infinite version of the elementary abelian groups from GT). The elements $\overline{x} := xN_X$ with $x \in X$ form a generating set of $\overline{F_X}$. Suppose there are pairwise distinct $x_1,\dots,x_n \in X$ with $x_1\dots x_n \in N_X$. Let $\sigma\colon X \to \mathbb{F}_2$ with $\sigma(x_1) = 1$ and $\sigma(x_i) = 0$ for $i = 2,\dots,n$. Let $\widehat{\sigma}\colon F_X \to \mathbb{F}_2$ be the extension of σ. Because of $\widehat{\sigma}(g^2) = 2\widehat{\sigma}(g) = 0$ and $\widehat{\sigma}([g,h]) = \widehat{\sigma}(g) + \widehat{\sigma}(h) - \widehat{\sigma}(g) - \widehat{\sigma}(h) = 0$, it follows that $x_1\dots x_n \in N_X \subseteq \mathrm{Ker}(\widehat{\sigma})$. This contradicts $\widehat{\sigma}(x_1\dots x_n) = \sigma(x_1) = 1$. Thus $x_1\dots x_n \notin N_X$. This shows that $\{\overline{x} : x \in X\}$ is a basis of $\overline{F_X}$. Thus it follows that $|X| = \dim \overline{F_X} = \dim \overline{F_Y} = |Y|$.[4]

(ii) Follows as in (iii) (only the universal property is used). □

Definition 1.6. If F is free w.r.t. X, then $\mathrm{rk}\, F := |X|$ is called the *rank* of F. According to Theorem 1.5, there is up to isomorphism only one free group of rank $r \in \mathbb{N}$. We denote this by F_r.

Lemma 1.7 (cf. GT-Exercise 60)**.** *Every word $w \in W$ is equivalent to exactly one reduced word $\tilde{w} \in W$.*

Proof (Van der Waerden). We already know that w is equivalent to at least one reduced word. For uniqueness, let $R \subseteq W$ be the set of all reduced words. For $r = x_1^{\epsilon_1}\dots x_n^{\epsilon_n} \in R$ and $x \in X$ let

$${}^x r := \begin{cases} x x_1^{\epsilon_1}\dots x_n^{\epsilon_n} & \text{if } x \neq x_1^{-\epsilon_1}, \\ x_2^{\epsilon_2}\dots x_n^{\epsilon_n} & \text{if } x = x_1^{-\epsilon_1}. \end{cases}$$

Obviously x induces a permutation $\sigma(x) \in \mathrm{Sym}(R)$ with inverse map $\sigma(x^{-1})$. By the universal property, σ extends to an action $F_X \to \mathrm{Sym}(R)$. For equivalent reduced words $v, w \in R$ it holds that

$$v = {}^{[v]}1 = {}^{[w]}1 = w. \qquad \square$$

Remark 1.8.

(i) For $r \geq 2$, F_r is non-abelian, because $xyx^{-1}y^{-1} \neq 1$ is reduced for $x \neq y$.

(ii) Let $w = x_1^{\epsilon_1}\dots x_n^{\epsilon_n} \in W$ be reduced with finite order k in F_X. By conjugating with $x_1^{-\epsilon}$ if necessary, one can assume $x_1^{\epsilon_1} \neq x_n^{-\epsilon_n}$. Then w^k is also reduced and it follows that $w = 1$. Thus F_r is torsion-free.

Corollay 1.9. *If F is free w.r.t. $X \subseteq F$, then every element of F can be uniquely written in the form $x_1^{a_1}\dots x_n^{a_n}$, where $n \in \mathbb{N}_0$, $x_1,\dots,x_n \in X$ with $x_i \neq x_{i+1}$ for $i = 1,\dots,n-1$ and $a_1,\dots,a_n \in \mathbb{Z}\setminus\{0\}$. In particular, $F = \langle X\rangle$.*

[4] requires the axiom of choice if $|X| = \infty$

Proof. Wlog. let $F = F_X$. Every element can be uniquely written in reduced form according to Lemma 1.7. By collecting identical letters, one obtains a representation in the desired form. □

Remark 1.10. According to Cayley, every group is isomorphic to a subgroup of a symmetric group. Free groups possess a dual property.

Theorem 1.11. *Every group G is isomorphic to a factor group of a free group F. If G can be generated by n elements, then one can choose* $\operatorname{rk} F = n$.

Proof. Let X be a generating set of G (if necessary $X = G$). Then the inclusion $X \to G$ can be extended to an epimorphism $F_X \to G$. The claim follows from the homomorphism theorem. □

Example 1.12. Let

$$a := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

We show that $G := \langle a, b \rangle \leq \mathrm{GL}(2, \mathbb{Q})$ is free w.r.t. $\{a, b\}$. For this, let $X = \{x, y\}$ and $\varphi\colon F_X \to G$ be the epimorphism with $\varphi(x) = a$ and $\varphi(y) = b$. Suppose there exists a non-trivial reduced word $w := z_1^{k_1} \dots z_n^{k_n} \in \operatorname{Ker}(\varphi)$ with $z_1, \dots, z_n \in X$ and $k_1, \dots, k_n \in \mathbb{Z} \setminus \{0\}$. After conjugation, we can assume $z_n = x$ and $k_n > 0$. Then $1 = \varphi(w) = \dots a^{k_{n-2}} b^{k_{n-1}} a^{k_n}$ holds. For $k \in \mathbb{Z}$, we have

$$a^k := \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}, \qquad b^k := \begin{pmatrix} 1 & 0 \\ 2k & 1 \end{pmatrix}$$

(induction on k). Let

$$V_> := \big\{(s,t) \in \mathbb{Q}^2 : |s| > |t|\big\}, \qquad V_< := \big\{(s,t) \in \mathbb{Q}^2 : |s| < |t|\big\}.$$

For $v := (1,1) \in \mathbb{Q}^2$, we have $a^{k_n} v = (2k_n + 1, 1) \in V_>$ because $k_n > 0$. For $v = (s,t) \in V_>$, we have $b^{k_{n-1}} v = (s, 2k_{n-1}s + t) \in V_<$, since $|2k_{n-1}s + t| \geq 2|k_{n-1}||s| - |t| > |s|$ (triangle inequality) because $k_{n-1} \neq 0$. Analogously, $a^{k_{n-2}} v \in V_>$ for $v \in V_<$ etc. [5] This yields the contradiction

$$(1,1) = \varphi(w)(1,1) = \dots a^{k_{n-2}} b^{k_{n-1}} a^{k_n} (1,1) \in V_> \cup V_<.$$

Thus φ is injective and $G \cong F_X \cong F_2$.

Remark 1.13.

(i) Let X be a generating set for G and $\sigma\colon F_X \to G$ with $\sigma(x) = x$ as in Theorem 1.11. The elements in $\operatorname{Ker}(\sigma)$ are called *relators* for G w.r.t. X. For $x_1^{\epsilon_1} \dots x_n^{\epsilon_n} \in \operatorname{Ker}(\sigma)$, we thus have $x_1^{\epsilon_1} \dots x_n^{\epsilon_n} = 1$ in G. An equation of this form is called a *relation* for G w.r.t. X.

(ii) Conversely, let $R \subseteq F_X$. Let $N := \langle R \rangle^{F_X} := \langle gRg^{-1} : g \in F_X \rangle \trianglelefteq F_X$ be the normal closure of R in F_X. We set

$$G := \langle X \mid R \rangle = \langle X \mid \{r = 1 : r \in R\} \rangle := F_X / N.$$

One often identifies letters $x \in X$ with their cosets $xN \in G$ (in general not injective!). If $|X| + |R| < \infty$, then G is called *finitely presented*. In this way, every group can be described by generators and relations (this corresponds to the statement that every vector space is the solution set of a system of linear equations). In general, however, it is difficult to read off the properties of G from X and R.

[5] This argument is called the *Ping-Pong Lemma*.

Example 1.14.

(i) $\langle X \mid \varnothing \rangle \cong F_X$.

(ii) $\langle x \mid x^n \rangle = \langle x \mid x^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z} \cong C_n$.

(iii) Every finite group G is finitely presented: Let $X := \{x_g : g \in G\}$, $R := \{x_g x_h x_{gh}^{-1} : g, h \in G\}$ and $N := \langle R \rangle^{F_X}$. Then there exists an epimorphism $\varphi\colon F_X \to G$ with $\varphi(x_g) = g$ and $R \subseteq \mathrm{Ker}(\varphi)$. Because of $\mathrm{Ker}(\varphi) \trianglelefteq F_X$, it follows that $N \subseteq \mathrm{Ker}(\varphi)$. Conversely, let $w := x_{g_1}^{\epsilon_1} \dots x_{g_n}^{\epsilon_n} \in \mathrm{Ker}(\varphi)$. Because of $x_1 = x_1 x_1 x_1^{-1} \in R$ and $x_g x_{g^{-1}} x_1^{-1} \in R$, we have $x_g x_{g^{-1}} \in \langle R \rangle$ and $x_g^{-1} \equiv x_{g^{-1}} \pmod N$. It follows that

$$w \equiv x_{g_1}^{\epsilon_1} \dots x_{g_n}^{\epsilon_n} \equiv x_{g_1^{\epsilon_1} g_2^{\epsilon_1}} x_{g_3^{\epsilon_3}} \dots \equiv x_{g_1^{\epsilon_1} \dots g_n^{\epsilon_n}} \equiv x_1 \equiv 1 \pmod N.$$

Thus $\mathrm{Ker}(\varphi) \subseteq N$ and $G = \langle X \mid R \rangle$.

Theorem 1.15 (von Dyck)**.** *Let $G = \langle x_i : i \in I \rangle$ and $H = \langle y_i : i \in I \rangle$ be groups, such that for every relation $x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n} = 1$ in G, the relation $y_{i_1}^{\epsilon_1} \dots y_{i_n}^{\epsilon_n} = 1$ also holds in H. Then there exists an epimorphism $G \to H$ with $f(x_i) = y_i$ for $i \in I$.*

Proof. By the universal property, there exist epimorphisms $f_G\colon F_I \to G$ and $f_H\colon F_I \to H$ with $f_G(i) = x_i$ and $f_H(i) = y_i$ for $i \in I$. By assumption, $\mathrm{Ker}(f_G) \leq \mathrm{Ker}(f_H)$ holds. Thus

$$G \cong F_I/\mathrm{Ker}(f_G) \to (F_I/\mathrm{Ker}(f_G))/(\mathrm{Ker}(f_H)/\mathrm{Ker}(f_G)) \cong F_I/\mathrm{Ker}(f_H) \cong H$$

is the desired epimorphism. □

Remark 1.16. One can use von Dyck's theorem to approximate opaque group presentations by known groups.

Example 1.17.

(i) Let $G := \langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \ \forall i, j \rangle$. Obviously G is abelian and every element in G has the form $x_1^{a_1} \dots x_n^{a_n}$ with $a_1, \dots, a_n \in \mathbb{Z}$. Now let $H := \langle y_1 \rangle \oplus \dots \oplus \langle y_n \rangle \cong C_\infty^n$. By Theorem 1.15, there exists an epimorphism $f\colon G \to H$ with $f(x_i) = y_i$ for $i = 1, \dots, n$. Obviously f is also injective and $G \cong H \cong C_\infty^n$. This also shows $F_n/F_n' \cong C_\infty^n$.

(ii) Let $G = \langle x, y \rangle$ with $x \neq y$ and $|\langle x \rangle| = |\langle y \rangle| = 2$. Then G consists of the elements of the form $xyxy \dots$ and $yxyx \dots$. If G is finite, then $n := |\langle xy \rangle| \in \mathbb{N}$ holds and every element has the form $x^i (xy)^j$ with $0 \leq i \leq 1$ and $0 \leq j \leq n-1$. Then it follows that

$$G \cong D_{2n} := \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1, \ \tau\sigma\tau = \sigma^{-1} \rangle,$$

where $D_4 = C_2^2$.

Remark 1.18 (GAP)**.**

```
F:=FreeGroup("x","y");; #the double semicolon suppresses the output
AssignGeneratorVariables(F);;
G:=F/[x^2,y^3,(x*y)^5]; #finitely presented group
Size(G);
H:=Image(IsomorphismPermGroup(G)); #isomorphic permutation group (more efficient)
StructureDescription(H);

#The reverse way:
G:=Image(IsomorphismFpGroup(H));; #isomorphic finitely presented group
RelatorsOfFpGroup(G); #new relations

#Calculating in the free group:
x:=(1,2,3,4,5,6,7,8,9,10,11);; y:=(3,7,11,8)(4,10,5,6);; #permutations of S_11
G:=Group(x,y);;
epi:=GroupHomomorphismByImages(F,G,GeneratorsOfGroup(F),[x,y]);
z:=Random(G);
PreImagesRepresentative(epi,z); #representation of z as a word in x,y
```

Theorem 1.19 (Neumann). *Let G be finitely presented and X an arbitrary generating set of G. Then there exists a finite presentation $G = \langle X_0 \mid R\rangle$ with $X_0 \subseteq X$.*

Proof. Let $G = \langle y_1, \dots, y_n \mid s_1, \dots, s_m\rangle$ be a finite presentation. Each y_i can be expressed by finitely many $x \in X$, say $y_i = w_i(x)$. Therefore, there exists a finite subset $X_0 = \{x_1, \dots, x_k\} \subseteq X$ with $G = \langle X_0\rangle$. Conversely, the x_i can be expressed by y_j, say $x_i = v_i(y)$. One obtains the following relations in X_0:

$$s_i(w_1(x), \dots, w_n(x)) = 1, \qquad x_i = v_i(w_1(x), \dots, w_n(x)).$$

Let R be the set of these relators and $H := \langle X_0 \mid R\rangle$. By von Dyck, there exists an epimorphism $\varphi\colon H \to G$ with $\varphi(x_i) = x_i$ for $i = 1, \dots, k$. One can now define $y_i := w_i(x)$ in H. Because of $x_i = v_i(y_1, \dots, y_n)$, it follows that $H = \langle y_1, \dots, y_n\rangle$. Since the relations in s_i also hold in H, there exists an epimorphism $\psi\colon G \to H$ with $\psi(y_i) = y_i$ for $i = 1, \dots, n$. Because of

$$\psi(\varphi(x_i)) = \psi(x_i) = \psi(v_i(y)) = v_i(y) = x_i,$$
$$\varphi(\psi(y_i)) = \varphi(y_i) = \varphi(w_i(x)) = w_i(x) = y_i$$

φ and ψ are mutually inverse isomorphisms. □

Theorem 1.20 (Hall). *Let $N \trianglelefteq G$. If N and G/N are finitely presented, then so is G.*

Proof. Let $N = \langle x_1, \dots, x_n \mid r_1, \dots, r_m\rangle$ and $G/N = \langle y_1N, \dots, y_kN \mid s_1, \dots, s_l\rangle$. Certainly $G = \langle x_1, \dots, x_n, y_1, \dots, y_k\rangle$. Because of $s_i(y) \in N$, relations $s_i(y) = t_i(x)$ hold. The normal subgroup property can be expressed by relations $y_ix_jy_i^{-1} = u_{ij}(x)$ and $y_i^{-1}x_jy_i = v_{ij}(x)$. We define

$$H := \langle x_1, \dots, x_n, y_1, \dots, y_k \mid \forall i, j : r_i, s_i(y)t_i(x)^{-1}, u_{ij}(x)y_ix_j^{-1}y_i^{-1}, v_{ij}(x)y_i^{-1}x_j^{-1}y_i \ \rangle.$$

Then there exists an epimorphism $\varphi\colon H \to G$ with $\varphi(x_i) = x_i$ and $\varphi(y_j) = y_j$. Let $\tilde{N} := \langle x_1, \dots, x_n\rangle \leq H$. The relations u_{ij} and v_{ij} show $\tilde{N} \trianglelefteq H$. For $h \in \tilde{N} \cap \mathrm{Ker}(\varphi)$ it holds that $h \in \langle r_1, \dots, r_m\rangle^{F_X}$. This shows $h = 1$ and $\tilde{N} \cap \mathrm{Ker}(\varphi) = 1$. Obviously φ induces an epimorphism $\overline{\varphi}\colon H/\tilde{N} \to G/N$ with $\overline{\varphi}(y_i\tilde{N}) = y_iN$. The relations r_i, t_i, u_{ij} and v_{ij} become trivial in $H/\tilde{N}$. Thus $H/\tilde{N}$ and G/N satisfy the same relations ($s_i(y) = 1$) and $\overline{\varphi}$ is an isomorphism. For $h \in \mathrm{Ker}(\varphi)$ it holds that $h\tilde{N} \in \mathrm{Ker}(\overline{\varphi}) = 1$, so $h \in \tilde{N} \cap \mathrm{Ker}(\varphi) = 1$. Thus φ is an isomorphism. □

Theorem 1.21. *Let $G = \langle X \mid R\rangle$ with $X = \{x_1, \ldots, x_n\}$ and $R = \{r_1, \ldots, r_k\}$. For $i = 1, \ldots, n$ let $r_i \equiv x_1^{a_{i1}} \ldots x_n^{a_{in}} \pmod{F'_X}$. Let $d_1 \mid \ldots \mid d_l$ be the elementary divisors of $A = (a_{ij}) \in \mathbb{Z}^{n\times k}$, where $l := \min\{n, k\}$. Then $G/G' \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_l\mathbb{Z} \times \mathbb{Z}^{n-l}$. In particular, $|G| = |G/G'| = \infty$ if $k < n$.*

Proof. Let $F := F_X$. According to Example 1.17, F/F' is a free abelian group of rank n. For $N := \langle rF' : r \in R\rangle \leq F/F'$ it now holds that $G/G' \cong (F/F')/N \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_s\mathbb{Z}$. □

Remark 1.22.

(i) According to Theorem 1.21, one can algorithmically decide whether a finitely presented group is perfect (nevertheless, one does not know whether the group is trivial).

(ii) Only few *finite* groups $G = \langle X \mid R\rangle$ with $|X| = |R|$ are known (for example Q_{2^n} according to Exercise 2).[6] It is even conjectured that every such group can be generated by three elements. This was proven for p-groups (cf. Theorem 5.25 and Exercise 22).

Example 1.23. Let $G = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1\rangle$. Then one obtains the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 7 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus $G = G'$ is perfect. One can verify that the matrices

$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

in $\mathrm{GL}(3, 2)$ satisfy the relations. Thus $G \neq 1$ holds. We will show in Theorem 10.50 that G is infinite.

Lemma 1.24. *Let N be the normal closure of $Y \subseteq X$ in F_X. Then F_X/N is free with respect to $X \setminus Y$.*

Proof. Wlog. let $\varnothing \neq Y \subset X$. Every element of N is a product of elements of the form gwg^{-1} with $w \in \langle Y\rangle$ and $g \in F_X$. The sum of the exponents of a letter $x \in X \setminus Y$ in gwg^{-1} is 0, because x can only occur in g and g^{-1}. This shows that the map $\sigma\colon X \setminus Y \to F/N$, $x \mapsto xN$ is injective. We show that F_X/N is free with respect to $\sigma(X \setminus Y)$. Let G be an arbitrary group and $\alpha\colon X \setminus Y \to G$ a function. If one sets

$$\overline{\alpha}(x) := \begin{cases} 1 & \text{if } x \in Y \\ \alpha(x) & \text{if } x \notin Y \end{cases}$$

for $x \in X$, one obtains an extension $\overline{\alpha}\colon X \to G$ of α. Since F is free with respect to X, there exists a homomorphism $\overline{\beta}\colon F \to G$ with $\overline{\beta}(x) = \overline{\alpha}(x)$ for all $x \in X$. Obviously, $N \leq \mathrm{Ker}(\overline{\beta})$ then holds. Therefore, there exists a homomorphism $\beta\colon F/N \to (F/N)/(\mathrm{Ker}(\overline{\beta})/N) \to F/\mathrm{Ker}(\overline{\beta}) \to G$ with $\beta(\sigma(x)) = \alpha(x)$ for all $x \in X \setminus Y$. Now let $\beta'\colon F/N \to G$ be another homomorphism with $\beta'(\sigma(x)) = \alpha(x)$ for all $x \in X \setminus Y$. Because of

$$F/N = \langle xN : x \in X\rangle = \langle xN : x \in X \setminus Y\rangle = \langle \sigma(x) : x \in X \setminus Y\rangle$$

it then follows immediately that $\beta = \beta'$. □

[6] One speaks of a *balanced* presentation.

Remark 1.25.

(i) The *Cayley graph* $\Omega(G, X)$ of a finitely generated group $G = \langle X \mid R \rangle$ is a graph with vertex set G, such that $g, h \in G$ form an edge if and only if $g^{-1}h \in X \cup X^{-1}$ (by $g^{-1}h \in X$ one obtains a directed graph). Obviously, $\Omega(G, X)$ depends on X. To avoid loops, we assume $1 \notin X$. In any case, $\Omega(G, X)$ is connected and regular (i. e. all vertices have the same number of edges).

(ii) A cycle in $\Omega(G, X)$ corresponds to a reduced word $x_1^{\epsilon_1} \dots x_n^{\epsilon_n} = 1$. Therefore, $\Omega(G, X)$ is a tree (i. e. acyclic) if and only if G is free with respect to X.

(iii) The group G permutes the vertices and edges of $\Omega(G, X)$ by left multiplication. Therefore, G is a subgroup of $\mathrm{Aut}(\Omega(G, X))$. On this basis, it can be shown that every finite group is the automorphism group of a graph (FRUCHT's theorem[7]).

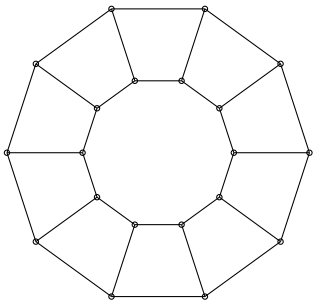
(iv) According to the Euler-Hierholzer theorem, $\Omega(G, X)$ is Eulerian if and only if $|X \cup X^{-1}|$ is even. This means there is a closed path that visits every edge of $\Omega(G, X)$ exactly once. The open Lovász conjecture states that $\Omega(G, X)$ is Hamiltonian for $2 < |G| < \infty$. This means there is a closed path that visits every vertex of $\Omega(G, X)$ exactly once.

(v) If G is finite, the eigenvalues of the adjacency matrix of $\Omega(G, X)$ can be investigated. It can be shown that they are closely related to the values of the complex irreducible characters of G.

Example 1.26.

(i) The Cayley graph of $G = \langle x \mid x^n \rangle$ w.r.t. $X = \{x\}$ is an n-gon. If one chooses $X = G$, one obtains the complete graph with n vertices.

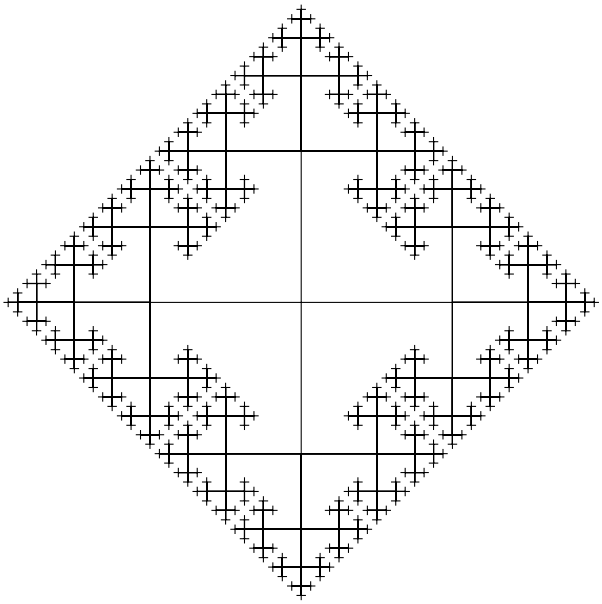
(ii) Let $G = D_{2n}$ be generated by two reflections $X = \{x, y\}$. Then $\Omega(G, X)$ is a $2n$-gon just like for C_{2n}. The Cayley graph thus does not determine whether G is abelian. If one chooses instead $X = \{x, z\}$ with a rotation z by $360°/n$, one obtains (with $n = 10$):



The two circles correspond to the cosets $\langle z \rangle$ and $x\langle z \rangle$.

[7] See Algebra notes

(iii) The Cayley graph of F_2 can be drawn (approximately) as a fractal:



Interactive examples can be found at `https://juliapoo.github.io/Cayley-Graph-Plotting/`.

(iv) Let $G = \langle X \rangle$ be the group of the $3 \times 3 \times 3$ Rubik's Cube and X the set of 90° and 180° rotations of the six faces. Then *God's Number* (20) is the diameter of $\Omega(G, X)$, i.e., the maximal length of a cycle-free path (see Christmas lecture Group Theory).

Remark 1.27 (GAP).

```
LoadPackage("grape",false);
G:=AlternatingGroup(5);;
Y:=[(1,2,3),(3,4,5)];; #X is read-only
C:=CayleyGraph(G,Y); #replaces Y by Y∪Y^-1
Diameter(C);

LoadPackage("hap",false); #loads further packages
G:=DihedralGroup(22);;
Y:=GeneratorsOfGroup(G);
CayleyGraphOfGroupDisplay(G,Y,"chromium"); #display in browser chromium, requires GraphViz
```

2 Subgroups of free groups

Remark 2.1. We already know that every group is a subgroup (resp. factor group) of a symmetric (resp. free) group. The factor groups of a finite symmetric group are themselves symmetric (since A_n is simple for $n \geq 5$). Dually to this, we show that the subgroups of a free group are themselves free. We start with a variation of the universal property.

Lemma 2.2 (Steinberg). *A group F is free w.r.t. $X \subseteq F$ if and only if for every non-empty set Ω and every map $\sigma\colon X \to \mathrm{Sym}(\Omega)$ there exists exactly one action $F \to \mathrm{Sym}(\Omega)$ with ${}^x\omega = \sigma(x)(\omega)$ for all $x \in X$ and $\omega \in \Omega$.*

Proof. If F is free w.r.t. X, then σ can be extended to an action with the specified property according to the universal property. For the converse, we first show $F = \langle X \rangle$. The map $\sigma\colon X \to \mathrm{Sym}(\langle X \rangle)$ by left multiplication (i.e., $\sigma(x)(y) = xy$) can be extended to an action $\rho\colon F \to \mathrm{Sym}(\langle X \rangle)$. However, one can also consider $\sigma\colon X \to \mathrm{Sym}(F)$ and, due to uniqueness, obtains exactly the same extension $\rho\colon F \to \mathrm{Sym}(F)$. Since the action by left multiplication is transitive, it follows that $\langle X \rangle = F$.

Now let a group G and a map $\sigma\colon X \to G$ be given. Let $\tau\colon G \to \mathrm{Sym}(G)$ be the monomorphism from Cayley's Theorem. By assumption, there exists exactly one action $\rho\colon F \to \mathrm{Sym}(G)$ with $\rho(x) = \tau(\sigma(x))$ for all $x \in X$. Here, $\rho(F) = \rho(\langle X \rangle) \leq \tau(G)$ holds. Clearly, $\tau^{-1}\rho\colon F \to G$ is a homomorphism that extends σ. If $\gamma\colon F \to G$ is also an extension of σ, then $\tau\gamma = \rho$ by assumption. This shows $\gamma = \tau^{-1}\rho$. □

Definition 2.3. Let F be free w.r.t. $X \subseteq F$ and $G \leq F$. A *Schreier transversal* for G is a system of representatives S for F/G with the following property: If $x^\epsilon s \in S$ is reduced with $x \in X$ and $\epsilon = \pm 1$, then $s \in S$ also holds.

Lemma 2.4. *Every subgroup of a free group possesses a Schreier transversal.*

Proof. Let F be free w.r.t. $X \subseteq F$ and $G \leq F$. Let the *length* of a coset aG be the minimal length of a reduced word in aG. We construct a Schreier transversal S of G by induction on the length of the cosets. Obviously, $1G$ is the only coset with length 0. Thus, let $1 \in S$. Now let aG have length $l \geq 1$. Suppose a has length l. Let $a = x^\epsilon b$ be reduced with $x \in X$ and $\epsilon = \pm 1$. By induction, there exists $s \in S$ with $sG = bG$. We choose $x^\epsilon s \in S$ as the representative of aG. Clearly, a Schreier transversal is formed in this way. □

Theorem 2.5 (Nielsen-Schreier)**.** *Subgroups of free groups are free.*

Proof. Let F be free w.r.t. $X \subseteq F$ and $G \leq F$. Let T be a Schreier transversal of G. For $a \in F$ let $\overline{a} \in T$ with $aG = \overline{a}G$. We show using Lemma 2.2 that G is free w.r.t.

$$Y := \{(\overline{xt})^{-1}xt : (x,t) \in (X,T),\ xt \neq \overline{xt}\} \subseteq G.$$

For this, let $\sigma\colon Y \to \mathrm{Sym}(\Omega)$ be an arbitrary map. We extend σ by the rule $\sigma(1) := \mathrm{id}_\Omega$. The map $f\colon X \to \mathrm{Sym}(\Omega \times T)$ with $f(x)(\omega,t) = (\sigma((\overline{xt})^{-1}xt)(\omega), \overline{xt})$ extends to an action $\widehat{f}\colon F \to \mathrm{Sym}(\Omega \times T)$. We show that the restriction $\widehat{\sigma}\colon G \to \mathrm{Sym}(\Omega \times \{1\})$ of $\widehat{f}$ is the unique homomorphic extension of σ. We first show ${}^t(\omega,1) = (\omega,t)$ for all $t \in T$. This is clear for $t = 1$. So let $t = x^\epsilon s$ be reduced with $x \in X$, $\epsilon = \pm 1$. Since T is a Schreier transversal, $s \in T$ holds. By induction on the length of t we can assume ${}^s(\omega,1) = (\omega,s)$. In the case $\epsilon = 1$ it follows that

$${}^t(\omega,1) = {}^x(\omega,s) = (\sigma((\overline{xs})^{-1}xs)(\omega),t) = (\sigma(1)(\omega),t) = (\omega,t).$$

Now let $\epsilon = -1$. Then ${}^x(\omega,t) = (\sigma((\overline{xt})^{-1}xt)(\omega),xt) = (\omega,s)$ and

$${}^t(\omega,1) = {}^{x^{-1}s}(\omega,1) = {}^{x^{-1}}(\omega,s) = (\omega,t).$$

Now let $y := (\overline{xt})^{-1}xt \in Y$ be arbitrary. Then

$${}^{xt}(\omega,1) = {}^x(\omega,t) = (\sigma(y)(\omega),\overline{xt}) = {}^{\overline{xt}}(\sigma(y)(\omega),1)$$

and $\widehat{\sigma}(y) = \sigma(y)$. Thus $\widehat{\sigma}$ is an extension of σ. Let $\tau\colon G \to \mathrm{Sym}(\Omega)$ also be an extension. Then $\rho\colon F \to \mathrm{Sym}(\Omega \times T)$ with $\rho(a)(\omega, t) := (\tau((\overline{at})^{-1}at)(\omega), \overline{at})$ is an action, because for $a, b \in F$ and $t \in T$ we have

$$\begin{aligned}(\rho(a)\rho(b))(\omega, t) &= \rho(a)(\tau((\overline{bt})^{-1}bt)(w), \overline{bt}) = (\tau((\overline{a\overline{bt}})^{-1}a\overline{bt})(\tau((\overline{bt})^{-1}bt)(w)), \overline{a\overline{bt}}) \\ &= (\tau((\overline{abt})^{-1}abt)(\omega), \overline{abt}) = \rho(ab)(\omega, t).\end{aligned}$$

For $x \in X$ we have

$$\rho(x)(\omega, t) = (\tau((\overline{xt})^{-1}xt)(\omega), \overline{xt}) = (\sigma((\overline{xt})^{-1}xt)(\omega), \overline{xt}) = {}^{x}(\omega, t).$$

Because of $F = \langle X \rangle$, ρ thus coincides with the action defined above. In particular, $\tau = \widehat{\sigma}$. □

Remark 2.6. For vector spaces (or free abelian groups) $U \leq V$ it is well known that $\dim U \leq \dim V$. For free groups this is completely false.

Theorem 2.7 (Schreier's Formula)**.** *If F is free and $G \leq F$ with $|F : G| < \infty$, then*

$$\boxed{\mathrm{rk}(G) = |F : G|(\mathrm{rk}(F) - 1) + 1.}$$

Proof. It suffices to determine the cardinality of the set Y in Theorem 2.5. Let $s, t \in T$ and $x, y \in X$ with

$$(\overline{xt})^{-1}xt = (\overline{ys})^{-1}ys \neq 1.$$

Then $xt, ys \notin T$. Since T is a Schreier transversal, xt and ys must be reduced. Suppose $\overline{xt} = xt'$ is reduced. Then it follows $t' \in T$ with

$$t'G = x^{-1}xt'G = x^{-1}\overline{xt}G = x^{-1}xtG = tG,$$

thus $t = t'$. This contradicts the choice of x and t. Therefore, $\overline{xt}$ does not begin with x. Consequently, x cannot cancel out from $(\overline{xt})^{-1}xt$. Analogously, y does not cancel out from $(\overline{ys})^{-1}ys$. Since xt and ys are reduced, xt must occur at the end of ys (or ys at the end of xt). If xt were actually shorter than ys, then xt would already occur in s and one obtains the contradiction $xt \in T$. Thus $xt = ys$ and it follows $(x, t) = (y, s)$. The listed elements of Y are therefore pairwise distinct.

Now let $t \in T \setminus \{1\}$. Let $t = x^{\epsilon}s$ in reduced form with $x \in X$ and $\epsilon = \pm 1$. Since T is a Schreier transversal, $s \in T$ holds. In the case $\epsilon = 1$, $xs \in T$ and otherwise $xt \in T$. Therefore, each $t \neq 1$ determines exactly one pair $(x', t') \in X \times T$ with $x't' \in T$. Conversely, every such pair arises in this way. This yields

$$|Y| = |X \times T| - |T \setminus \{1\}| = |T|(|X| - 1) + 1 = |F : G|(\mathrm{rk}(F) - 1) + 1.$$ □

Example 2.8. Let $F_2 = \langle x, y \rangle$ and let N be the normal closure of $\{x^2, y^2, (xy)^2\}$ in F_2. According to Example 1.17, F_2/N is the Klein four-group. According to Schreier's formula, $\mathrm{rk}\, N = 4(2-1)+1 = 5$. We choose the Schreier transversal $\{1, x, y, xy\}$ of N in F_2 and calculate

$$\begin{gathered}(\overline{xx})^{-1}xx = x^2,\ (\overline{xxy})^{-1}xxy = y^{-1}x^2y,\ (\overline{yy})^{-1}yy = y^2, \\ (\overline{yx})^{-1}yx = y^{-1}x^{-1}yx,\ (\overline{yxy})^{-1}yxy = x^{-1}yxy\end{gathered}$$

Thus N is free with respect to $\{x^2, y^2, y^{-1}x^2y, y^{-1}x^{-1}yx, x^{-1}yxy\}$.

Corollay 2.9. *The group F_2 possesses (free) subgroups of every finite or countable rank. For every countable group G there exist $N \trianglelefteq H \leq F_2$ with $G \cong H/N$ (cf. Theorem 11.25).*

Proof. Let $F_2 = \langle x, y\rangle$. Then $G := \langle x\rangle \leq F_2$ with $\operatorname{rk} G = 1$ (Schreier's formula does not apply here because of $|F : G| = \infty$). For $n \in \mathbb{N}$ there exists $N \trianglelefteq F$ with $F/N \cong C_n$ according to Theorem 1.11. Schreier's formula shows $\operatorname{rk} N = n + 1$. Finally, let $N := F' = [F, F]$. According to Example 1.17, $F/N \cong \mathbb{Z}^2$. Therefore, the elements $x^a y^b$ with $a, b \in \mathbb{Z}$ form a Schreier transversal of N in F. The set Y constructed in the proof of Theorem 2.5 includes the pairwise distinct reduced words $(\overline{yxy^b})^{-1} yxy^b = y^{-b-1}x^{-1}yxy^b$ with $b \in \mathbb{Z}$. Therefore, $\operatorname{rk} N = \infty$. The second statement follows from Theorem 1.11. □

Remark 2.10.

(i) The proof of Corollay 2.9 shows that not every subgroup of a finitely generated group must be finitely generated ($\operatorname{rk}(F_2') = \infty$).

(ii)
```
F:=FreeGroup("x","y");;
AssignGeneratorVariables(F);;
H:=Subgroup(F,[x^2,y^3,(x*y)^5]);;
IsFreeGroup(H);
Rank(H); #= 3
FreeGeneratorsOfGroup(H);
Index(F,H); #= ∞
N:=NormalClosure(F,H);;
Index(F,N); #= 60
GeneratorsOfGroup(N);; #speeds up next command
Rank(N);
```

Theorem 2.11. *If G is finitely generated and $H \leq G$ with $|G : H| < \infty$, then H is also finitely generated.*

Proof. Let $X = X^{-1}$ be a finite generating set of G and R a transversal for G/H with $1 \in R$. For $x \in X$ and $r \in R$ there exist $\alpha(x, r) \in H$ and $\gamma(x, r) \in R$ with $xr = \gamma(x, r)\alpha(x, r)$. Every element in H has the form $h = x_1 \dots x_n$ with $x_1, \dots, x_n \in X$. It holds that

$$\begin{aligned} h &= x_1 \dots x_n 1 = x_1 \dots x_{n-1}\gamma(x_n, 1)\alpha(x_n, 1) = x_1 \dots x_{n-2}\gamma(x_{n-1}, \gamma(x_n, 1))\alpha(x_{n-1}, \gamma(x_n, 1))\alpha(x_n, 1) \\ &= \dots = \gamma(x_1, \dots)\alpha(x_1, \dots) \dots \alpha(x_n, 1). \end{aligned}$$

Because $h \in H$, it follows that $\gamma(x_1, \dots) = 1$. Thus $H = \langle \alpha(x, r) : x \in X, r \in R\rangle$. □

Remark 2.12. The proof of Theorem 2.11 shows that one can generate H with $|G : H||X|$ elements. The next theorem gives an optimal estimate.

Theorem 2.13 (Reidemeister-Schreier)**.** *Let $G = \langle X \mid R\rangle$ be a group and $H \leq G$. Then a presentation $H = \langle Y \mid S\rangle$ can be derived from X and R. In the case $|G : H| < \infty$, it holds that $|Y| \leq |G : H|(|X| - 1) + 1$ and $|S| \leq |G : H||R|$.*

Proof. Let $F := F_X$ and $N := \langle R\rangle^F \trianglelefteq F$. Let $\varphi\colon F \to G$ be the epimorphism with $\operatorname{Ker}(\varphi) = N$. Let $\tilde{H} = \varphi^{-1}(H) \leq F$. Then $|F : \tilde{H}| = |F/N : \tilde{H}/N| = |G : H|$. Let T be a Schreier transversal of $\tilde{H}$ in F. As in the proof of Theorem 2.5, $\tilde{H}$ is free with respect to

$$Y := \{y_{xt} := (\overline{xt})^{-1}xt : (x, t) \in (X, T), y_{xt} \neq 1\} \subseteq \tilde{H}.$$

In particular, $\langle\varphi(Y)\rangle = \varphi(\tilde{H}) = H$. Additionally, let

$$y_{x^{-1}t} = (\overline{x^{-1}t})^{-1}x^{-1}t = (t^{-1}x\overline{x^{-1}t})^{-1} = y^{-1}_{x,\overline{x^{-1}t}}.$$

For a reduced word $w := x_1^{\epsilon_1}\dots x_n^{\epsilon_n} \in F$, let

$$\psi(w) := y_{x_n^{\epsilon_n},\overline{x_1^{\epsilon_1}\dots x_{n-1}^{\epsilon_{n-1}}}} y_{x_{n-1}^{\epsilon_{n-1}},\overline{x_1^{\epsilon_1}\dots x_{n-2}^{\epsilon_{n-2}}}} \dots y_{x_2^{\epsilon_2},\overline{x_1^{\epsilon_1}}} y_{x_1^{\epsilon_1},1} \in \tilde{H}$$

be a word in Y. Let $S := \{\psi(t^{-1}rt) : t \in T, r \in R\}$. We show $H \cong \langle Y \mid S\rangle$. For this, let $M := \langle S\rangle^{\tilde{H}} \trianglelefteq \tilde{H}$. First, we verify $\psi(w) = \overline{w}^{-1}w$ for $w \in F$ by induction on $|w|$. This is clear for $|w| \le 1$. Now let $w = zx^{\epsilon}$ be reduced. Then

$$\psi(w) = y_{x^{\epsilon},\overline{z}}\psi(z) = (\overline{x^{\epsilon}z})^{-1}x^{\epsilon}\overline{z}\cdot\overline{z}^{-1}z = \overline{w}^{-1}w.$$

For $h \in \tilde{H}$, it follows that $\psi(h) = (\overline{h})^{-1}h = h$. In particular, $\psi(t^{-1}rt) = t^{-1}rt \in N$ for $r \in R$ and $t \in T$. Thus $S \subseteq N$ and $M \subseteq N$ because $N \trianglelefteq \tilde{H}$. For the reverse inclusion, it suffices to show $g^{-1}rg \in M$ for $g \in F$ and $r \in R$. Let $g = th$ with $t \in T$ and $h \in \tilde{H}$. Then $g^{-1}rg = h^{-1}t^{-1}rth = h^{-1}\psi(t^{-1}rt)h \in M$. Thus $H \cong \tilde{H}/N = \tilde{H}/M = \langle Y \mid S\rangle$.

The second assertion follows from Schreier's formula and the construction of S. □

Remark 2.14 (GAP)**.**

```
G:=SymmetricGroup(6);;
FG:=Image(IsomorphismFpGroup(G));;
FH:=DerivedSubgroup(FG);; #= A_6
P:=PresentationSubgroup(FG,FH,"y"); #Presentation with generators y_i
TzPrintPresentation(P); #Statistics about generators and relations
```

Theorem 2.15 (Coxeter-Todd algorithm)**.** *Let $G = \langle X \mid R\rangle$ be finitely presented and $H \le G$ with $|G : H| < \infty$. Then there exists an algorithm that determines the action of G on G/H. In particular, $|G : H|$ can be calculated from a generating set of H.*

Sketch of proof. By Reidemeister-Schreier, there exists a finite generating system Y of H. Let $G/H = \{H = H_1, \dots, H_n\}$. For $y = x_1^{\epsilon_1}\dots x_k^{\epsilon_k} \in Y$ let $t_y := (t_1, \dots, t_k)$ with $x_i^{\epsilon_i}\dots x_k^{\epsilon_k}H = H_{t_i}$ for $i = 1, \dots, k$. Because of $y \in H$, $t_1 = 1$. For $r = x_1^{\epsilon_1}\dots x_l^{\epsilon_l} \in R$ let $T_r = (t_{ij}) \in \mathbb{N}^{n\times l}$ with $x_m^{\epsilon_m}\dots x_l^{\epsilon_l}H_i = H_{t_{im}}$ for $m = 1, \dots, l$. Because of $r = 1$ in G, $t_{i1} = i$ for $i = 1, \dots, n$. We fill the vectors t_y $(y \in Y)$ and matrices T_r $(r \in R)$ from left to right and from top to bottom by assigning new cosets and taking all logical consequences into account. For each new coset, a new row is added in T_r. Because of $|G : H| < \infty$, eventually all entries in t_y and T_r are filled. Furthermore, every $x \in X$ occurs in an $r \in R$ or in a generator of H. In this way, one can read off the action of G on G/H. Since G acts transitively, all cosets actually appear in the tables, d. h. one can read off $n = |G : H|$. □

Example 2.16. Let $G = \langle x, y \mid x^3 = y^5 = (xy)^2 = 1\rangle$ and $H := \langle y\rangle$. The vector $t_y = (1)$ contains no information here. The first row of T_{y^5} is $(1, 1, 1, 1, 1)$. The definition $H_2 := x^{-1}H_1 = x^2H$ yields the following entries in T_{x^3}, T_{y^5} and $T_{(xy)^2}$:

x	x	x
1	2	
2		

y	y	y	y	y
1	1	1	1	1
2				

x	y	x	y
1	2		1
2			

Now let $H_3 := x^{-1}H_2 = xH$. Then $x^{-1}H_3 = H_1$ and $y^4H_2 = y^{-1}x^{-1}H = xyH = H_3$.

x	x	x
1	2	3
2	3	1
3	1	2

y	y	y	y	y
1	1	1	1	1
2	3			
3				

x	y	x	y
1	2	3	1
2	3		
3	1	1	2

We further define $H_4 := y^{-1}H_3$, $H_5 := x^{-1}H_4$, $H_6 := y^{-1}H_4$, $H_7 := x^{-1}H_5$ etc.

x	x	x
1	2	3
4	5	7
6	9	8
10	11	12

y	y	y	y	y
1	1	1	1	1
2	3	4	6	5
7	8	10	11	9
12	12	12	12	12

x	y	x	y
1	2	3	1
2	3	4	5
5	7	8	6
6	9	7	4
9	8	10	11
11	12	12	10

(redundant rows were deleted). One can now read off $|G| = |G : H||H| = 12|H| \leq 60$. Indeed, the permutations $x = (1,2,3)$ and $y := (1,4,3,5,2)$ in A_5 satisfy the relations. By von-Dyck, $G \cong \langle (1,2,3),(1,4,3,5,2)\rangle = A_5$.

Remark 2.17 (GAP)**.**

```
F:=FreeGroup(2);;
G:=F/[F.1^3,F.2^5,(F.1*F.2)^2];; #F.n is the n-th generator
H:=Subgroup(G,[G.2]);;
CT:=CosetTable(G,H);;
Display(TransposedMat(CT)); #only columns F.1^±1 and F.2^±1
f:=FactorCosetAction(G,H); #action on G/H
StructureDescription(Image(f)); #corresponding permutation group

G:=F/[F.1^2,F.2^3,(F.1*F.2)^7];;
CT:=CosetTable(G,TrivialSubgroup(G));; #quits after 4096000 cosets (|G| = ∞)
```

For more complicated examples and better graphical implementation, one can use the packages `ACE` or `ICT` (requires `xgap`). On the cover, the table for $A_6 \leq M_{11}$ is given.

Theorem 2.18 (Moore)**.** *For $n \geq 2$, it holds that*

$$S_n \cong \langle x_1, \ldots, x_{n-1} \mid 1 = x_i^2 = (x_j x_{j+1})^3 = (x_k x_l)^2 \text{ for } k < l-1\rangle.$$

Proof. Let G be the group on the right side. For $n = 2$, $G \cong C_2 \cong S_2$ holds. So let $n > 2$ and $H := \langle x_1, \ldots, x_{n-2}\rangle \leq G$. By induction, H is a factor group of S_{n-1}. In particular, $|H| \leq (n-1)!$. We show that G permutes the following cosets:

$$H, x_{n-1}H, x_{n-2}x_{n-1}H, \ldots, x_1 \ldots x_{n-1}H.$$

Certainly,

$$x_i(x_i x_{i+1} \ldots x_{n-1}H) = x_{i+1} \ldots x_{n-1}H,$$
$$x_{i-1}(x_i x_{i+1} \ldots x_{n-1}H) = x_{i-1}x_i \ldots x_{n-1}H.$$

For $j < i-1$, $x_i x_j = (x_i x_j)^{-1} = x_j x_i$ holds and

$$x_j(x_i x_{i+1} \dots x_{n-1} H) = x_i x_{i+1} \dots x_{n-1} x_j H = x_i x_{i+1} \dots x_{n-1} H.$$

Now let $j > i$. Because of $(x_{j-1}x_j)^3 = 1$, $x_{j-1}x_j x_{j-1} = x_j x_{j-1} x_j$ holds. It follows that

$$\begin{aligned} x_j(x_i x_{i+1} \dots x_{n-1} H) &= x_i \dots x_{j-2}(x_j x_{j-1} x_j) x_{j+1} \dots x_{n-1} H = x_i \dots x_{j-2}(x_{j-1} x_j x_{j-1}) x_{j+1} \dots x_{n-1} H \\ &= x_i \dots x_{n-1} x_{j-1} H = x_i \dots x_{n-1} H. \end{aligned}$$

Since G generally operates transitively on G/H, $|G : H| \leq n$ and $|G| \leq n!$ holds.

Conversely, the transpositions $x_i' := (i, i+1) \in S_n$ for $i = 1, \dots, n-1$ satisfy the same relations and because of $S_n = \langle x_1', \dots, x_{n-1}' \rangle$, the assertion follows. □

Theorem 2.19 (Moore). *For $n \geq 2$, it holds that*

$$A_n \cong \langle x_1, \dots, x_{n-2} \mid 1 = x_1^3 = x_2^2 = \dots = x_{n-2}^2 = (x_i x_{i+1})^3 = (x_k x_l)^2 \text{ for } k < l-1 \rangle.$$

Proof. Let G again be the right side. For $n \leq 3$ the claim holds. Let $n \geq 4$ and $H := \langle x_1, \dots, x_{n-3} \rangle \leq G$. By induction $|H| \leq \frac{1}{2}(n-1)!$. We show that G permutes the following n cosets

$$H, x_{n-2}H, x_{n-3}x_{n-2}H, \dots, x_1 \dots x_{n-2}H, x_1^2 x_2 \dots x_{n-2}H.$$

As long as no x_1 is involved, this proceeds as in Theorem 2.18. For $i \geq 3$ we have $x_1 x_i = x_i x_1^{-1}$ and

$$\begin{aligned} x_1 x_i \dots x_{n-2} H &= x_i \dots x_{n-2} x_1^{\pm 1} H = x_i \dots x_{n-2} H, \\ x_i x_1^{\pm 1} x_2 \dots x_{n-2} H &= x_1^{\mp 1} x_i x_2 \dots x_{n-2} H = x_1^{\mp 1} x_2 \dots x_{n-2} H. \end{aligned}$$

From $(x_1 x_2)^3 = 1$ it follows that $x_2 x_1 x_2 = x_1^{-1} x_2 x_1^{-1}$ and

$$x_2 x_1^{\pm 1} \dots x_{n-2} H = x_1^{\mp 1} x_2 x_1^{\mp 1} x_3 \dots x_{n-2} H = x_1^{\mp 1} x_2 \dots x_{n-2} H.$$

As in Theorem 2.18 one obtains $|G| \leq n|H| \leq |A_n|$. Conversely, the elements $x_1 = (1,2,3)$, $x_i = (1,2)(i+1, i+2)$ $(i = 2, \dots, n-2)$ of A_n satisfy the given relations. □

Remark 2.20. Guralnick-Kantor-Kassabov-Lubotzky have shown that one can present all symmetric and alternating groups with two generators and eight relations (or three generators and seven relations[8]).

Definition 2.21. A reduced word $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ of a free group is called *cyclically reduced*, if $x_1^{\epsilon_1} \neq x_n^{-\epsilon_n}$.

Remark 2.22. For a group $\langle X \mid r \rangle$ with only one relation, one can always assume after conjugation that r is cyclically reduced. The following generalization of Lemma 1.24 now holds.

Theorem 2.23 (Magnus' Freiheitssatz). *Let $G = \langle X \mid r \rangle$, where r is cyclically reduced. If $x \in X$ occurs in r, then $\langle X \setminus \{x\} \rangle \leq G$ is free w.r.t. $X \setminus \{x\}$.*

Proof. See Theorem 7.1 in [Camps et al., *Einführung in die kombinatorische und geometrische Gruppentheorie*, Heldermann Verlag, Lemgo, 2008] □

[8] The presentation was flawed and was corrected by Huxford.

3 Automorphisms of free groups

Definition 3.1. Let $X \subseteq X'$ be alphabets and $R \subseteq F_X$ with normal closure $N \trianglelefteq F$. The transformations

- $(X, R) \to (X, R \cup \{r\})$ with $r \in N \setminus R$,
- $(X, R) \to (X \cup \{x\}, R \cup \{x^{-1}w\})$ with $x \in X' \setminus X$ and $w \in F_X$

and their inverse mappings are called *Tietze transformations*.

Theorem 3.2. *Two finitely presented groups $\langle X_1 \mid R_1\rangle$ and $\langle X_2 \mid R_2\rangle$ are isomorphic if and only if one can transform (X_1, R_1) into (X_2, R_2) by finitely many Tietze transformations.*

Proof. Wlog. let $X_1 \cap X_2 = \varnothing$. Let $F_i := F_{X_i}$ and $N_i := \langle R_i\rangle^{F_i} \trianglelefteq F_i$. For $r \in N_i$, it certainly holds that $\langle R_i \cup \{r\}\rangle^{F_i} = N_i$ and $G_i := \langle X_i \mid R_i\rangle \cong \langle X_i \mid R_i \cup \{r\}\rangle$. Now let $x \in X' \setminus X_i$ and $w \in F_i$. Then there exists an epimorphism $\varphi\colon F_{X_i \cup \{x\}} \to G_i$ with $\varphi(x_i) = x_i N_i$ for $x_i \in X_i$ and $\varphi(x) = wN_i$. Obviously,

$$N_i' := \langle R_i \cup \{x^{-1}w\}\rangle^{F_{X_i \cup \{x\}}} \subseteq \operatorname{Ker}(\varphi).$$

Conversely, let $y := y_1^{\epsilon_1} \dots y_n^{\epsilon_n} \in \operatorname{Ker}(\varphi)$ with $y_1, \dots, y_n \in X_i \cup \{x\}$. Let $k := |\{1 \le j \le n : y_j = x\}|$. If $k = 0$, then $y \in \operatorname{Ker}(\varphi) \cap F_i = N_i \subseteq N_i'$. Now let $k > 0$. To show $y \in N_i'$, we can assume $y_n^{\epsilon_n} = x$ after conjugation. It then suffices to show $yx^{-1}w \in N_i'$, but this follows by induction on k. Thus $\operatorname{Ker}(\varphi) = N_i'$ and $G_i \cong \langle X_i \cup \{x\} \mid R_i \cup \{x^{-1}w\}\rangle$. If $(X_1 \mid R_1)$ can be transformed into $(X_2 \mid R_2)$ by Tietze transformations, then G_1 and G_2 are isomorphic (for this, no finite presentation is needed).

Conversely, let $G := G_1 \cong G_2$. Then there exist epimorphisms $\varphi_i\colon F_i \to G$ with kernel N_i for $i = 1, 2$. Let $X := X_1 \cup X_2$ and $F := F_X$. Then there exists a homomorphism $\varphi\colon F \to G$ that extends φ_1 and φ_2. For $x \in X_1$, we choose $w_x \in F_2$ with $\varphi(x) = \varphi_2(w_x) = \varphi(w_x)$. Let $S_1 := \{s_x := x^{-1}w_x : x \in X_1\} \subseteq F$. We define S_2 analogously. By an obvious (finite) sequence of Tietze transformations, we transform (X_1, R_1) into $(X, R_1 \cup S_2)$. Certainly $N := \langle R_1 \cup S_2\rangle^F \subseteq \operatorname{Ker}(\varphi)$. As above, one shows $\operatorname{Ker}(\varphi) \subseteq N$. In particular, $R_2 \cup S_1 \subseteq N$. Thus, one can pass from $(X, R_1 \cup S_2)$ to $(X, R_1 \cup R_2 \cup S_1 \cup S_2)$ by Tietze transformations. The situation is now symmetric. Therefore, one can also pass from (X_2, R_2) to $(X, R_1 \cup R_2 \cup S_1 \cup S_2)$. The inverse Tietze transformations finally transform (X_1, R_1) into (X_2, R_2). □

Remark 3.3. The commands

`SimplifyPresentation` (= `TzGo`), `SimplifiedFpGroup`, `IsomorphismSimplifiedFpGroup`

implicitly perform Tietze transformations to simplify a finite presentation $\langle X \mid R\rangle$, i. e. $|X|$, $|R|$, and $\sum_{r \in R} l(r)$ are minimized. However, the transformations can also be applied specifically.

```
G:=PerfectGroup(768000,10); #perfect group from database
H:=Image(IsomorphismFpGroup(G));
P:=PresentationFpGroup(H,2); #presentation, print level=2, so more output
TzGoGo(P); #iterates simplifications with Tietze transformation
TzPrintPresentation(P); #5 generators, 33 relations, total length 224
gen:=SmallGeneratingSet(G); #also works with 2 generators
H:=Image(IsomorphismFpGroupByGenerators(G,gen)); #use these
P:=PresentationFpGroup(H,2);
TzGoGo(P);
TzPrintPresentation(P); #2 generators, 59 relations, total length 3220

gen:=GeneratorsOfPresentation(P);
TzSubstitute(P,gen[1]^2*gen[2]); #perform second Tietze transformation with w = x^2y
```

Definition 3.4. Let $F = F_X$ and $x, y \in X$ with $x \neq y$. The automorphisms

$$\alpha_x \colon F \to F, \qquad z \mapsto \begin{cases} x^{-1} & \text{if } z = x \\ z & \text{if } z \neq x \end{cases},$$

$$\beta_{xy} \colon F \to F, \qquad z \mapsto \begin{cases} xy & \text{if } z = x \\ z & \text{if } z \neq x \end{cases}$$

(where $z \in X$) are called *Nielsen transformations* (note the analogy to the Gaussian algorithm). Let $\mathrm{Aut}_N(F) := \langle \alpha_x, \beta_{xy} : x, y \in X, x \neq y \rangle \leq \mathrm{Aut}(F)$.

Remark 3.5. For distinct $x, y \in X$ we have

$$\begin{aligned} \alpha_y\beta_{xy}\alpha_x\beta_{yx}\alpha_y\beta_{xy}(x,y) &= \alpha_y\beta_{xy}\alpha_x\beta_{yx}\alpha_y(xy,y) = \alpha_y\beta_{xy}\alpha_x\beta_{yx}(xy^{-1},y^{-1}) \\ &= \alpha_y\beta_{xy}\alpha_x(y^{-1},x^{-1}y^{-1}) = \alpha_y\beta_{xy}(y^{-1},xy^{-1}) = \alpha_y(y^{-1},x) = (y,x). \end{aligned}$$

Thus, every permutation on X can be realized by Nielsen transformations.

Theorem 3.6. *Let $w_1, \ldots, w_n \in F = F_X$ and $\gamma \in \mathrm{Aut}(F)$. Then there exists a $\delta \in \mathrm{Aut}_N(F)$ with $\delta(w_i) = \gamma(w_i)$ for $i = 1, \ldots, n$. In particular, $\mathrm{Aut}(F) = \mathrm{Aut}_N(F)$ if $|X| < \infty$.*

Proof. Let $Y \subseteq X$ with $|Y| < \infty$ and $w_1, \ldots, w_n \in F_Y$. It suffices to construct a $\delta \in \mathrm{Aut}_N(F)$ with $\gamma(y) = \delta(y)$ for all $y \in Y$. Because of $\langle \gamma^{-1}(X) \rangle = F$, there exists a finite subset $Z \subseteq X$ with $Y \subseteq \langle \gamma^{-1}(Z) \rangle$. We can assume $Y \subseteq Z = \{x_1, \ldots, x_n\}$. Let $w_i := \gamma^{-1}(x_i)$ be reduced for $i = 1, \ldots, n$. Suppose there exist $1 \leq i, j \leq n$ with $|w_i w_j| < |w_i|$. Obviously then $i \neq j$. By replacing γ with $\beta_{x_i x_j}^{-1}\gamma$, w_i is replaced by $w_i w_j$, while w_k for $k \neq i$ remains unchanged. In this way, $\sum_{i=1}^n |w_i|$ becomes smaller. In the case $|w_i w_j| < |w_j|$, one can analogously perform the transformations $w_i \mapsto w_j \mapsto w_i w_j$ using Remark 3.5. We can therefore assume

$$|w_i w_j| \geq \max\{|w_i|, |w_j|\}$$

for $1 \leq i, j \leq n$. This means that at most half of w_i and w_j can cancel in the product $w_i w_j$. Now assume that i, j, k exist with $|w_i w_j w_k| \leq |w_i| - |w_j| + |w_k|$. Let $w_i = as^{-1}$, $w_j = sbt^{-1}$ and $w_k = tc$, such that $w_i w_j = abt^{-1}$ and $w_j w_k = sbc$ are reduced (note that only half of w_j can cancel on the left and right respectively). It holds that

$$|a| + |b| + |c| = |w_i w_j w_k| \leq |w_i| - |w_j| + |w_k| = |a| - |b| + |c|,$$

d. h. $b = 1$ and $|s| = |t|$. Because of $|w_i| = |a| + |s| = |w_i w_j| \geq |w_j| = 2|s|$, it holds that $|s| \leq \frac{1}{2}|w_i|$ and analogously $|s| \leq \frac{1}{2}|w_k|$. We can thus replace w_i by $w_i w_j$ or w_k by $w_j w_k$ (swapping w_j and w_k is feasible) without changing $\sum_{i=1}^n |w_i|$. For $w \in F_Z$, let $L(w)$ be the left subword of w of length $\lfloor (|w| + 1)/2 \rfloor$. Let $\leq$ be the lexicographical (well-)ordering on F_Z with

$$1 < x_n^{-1} < x_{n-1} < \ldots < x_1 < x_2 < \ldots < x_n < x_n^{-2} < x_n^{-1}x_{n-1}^{-1} < \ldots.$$

We write $w \prec v$ if $\min\{L(w), L(w^{-1})\} < \min\{L(v), L(v^{-1})\}$ or $(\min\{L(w), L(w^{-1})\} = \min\{L(v), L(v^{-1})\}$ and $\max\{L(w), L(w^{-1})\} < \max\{L(v), L(v^{-1})\})$, where the minimum/maximum is to be taken with respect to $\leq$. If $s < t$ in the above situation, then $w_j w_k = sc \prec tc = w_k$, because $L(c^{-1}s^{-1}) = L(c^{-1}) = L(c^{-1}t^{-1})$. If $t < s$, then $w_i w_j = at^{-1} \prec as^{-1} = w_i$. By suitable Nielsen transformations, we can thus achieve that the w_i are as small as possible with respect to $\prec$ (since $\leq$ is a well-ordering, the w_i cannot become arbitrarily small). The property $|w_i w_j| \geq \max\{|w_i|, |w_j|\}$ is preserved. In the end,

$$|w_i w_j w_k| > |w_i| - |w_j| + |w_k|$$

for $1 \leq i, j, k \leq n$, d. h. w_j does not cancel completely in $w_i w_j w_k$. The same procedure can be carried out more generally with the elements $w_i^{\pm 1}$.

Because of $Y \subseteq \langle \gamma^{-1}(Z) \rangle$, every $x_i \in Y$ can be represented as a product of the w_j, say $x_i = w_{j_1}^{\epsilon_1} \dots w_{j_k}^{\epsilon_k}$. By construction, however, $1 = |x_i| = |w_{j_1}^{\epsilon_1} \dots w_{j_k}^{\epsilon_k}| \geq k$, d. h. $x_i = w_{j_1}^{\epsilon_1}$. After further Nielsen transformations as in Remark 3.5, one finally achieves $w_i = x_i$ for $i = 1, \dots, n$. □

Remark 3.7.

(i) NIELSEN has given a finite presentation of $\mathrm{Aut}(F_n)$ with respect to the Nielsen transformations. NEWMAN has shown that one can generate $\mathrm{Aut}(F_n)$ with only two automorphisms (of infinite order).

(ii) In GAP one can construct Nielsen transformations as follows:

```
F:=FreeGroup("x","y");;
AssignGeneratorVariables(F);;
FreeGroupAutomorphismsGeneratorO(F); #Nielsen transformation α_x
FreeGroupAutomorphismsGeneratorU(F); #β_xy
FreeGroupAutomorphismsGeneratorP(F); #(x,y) ↦ (y,x)
A:=AutomorphismGroup(F);; #Aut(F)
iso:=IsomorphismFpGroup(A);; #Isomorphism from A to a finitely presented group
a:=GroupHomomorphismByImages(F,F,[x,y],[x^y,x*y]); #a ∈ A
a^iso; #a as a word in α_x, β_xy and (x,y) ↦ (y,x)
```

Theorem 3.8. *Let* $\Phi\colon \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$, *where* $\Phi(\alpha)_{ij}$ *is the exponent sum of* x_j *in* $\alpha(x_i)$. *Then* Φ *is an epimorphism.*

Proof. The map Φ arises from the restriction

$$\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/F_n') \cong \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n, \mathbb{Z})$$

and is therefore a well-defined homomorphism. For surjectivity, it suffices to show that every matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ is a product of $\Phi(\alpha_x)$ and $\Phi(\beta_{xy})$. Left multiplication (or right multiplication) by $\Phi(\alpha_x)$ causes a row (or column) of A to be multiplied by -1. Through $\Phi(\beta_{xy})$ one can add a row (or column) of A to another row (or column). As in Remark 3.5, one can also swap rows and columns of A. With these operations, A can be brought into Smith normal form. Because $\det A = \pm 1$, all elementary divisors are 1, i.e., A can be transformed into the identity matrix. □

4 Group Extensions

Remark 4.1. According to the Jordan-Hölder theorem, all finite groups (or those possessing a composition series) are built up from simple groups. The finite simple groups are, as is well known, completely classified. It remains to investigate how a group is composed of normal subgroups and factor groups.

Definition 4.2.

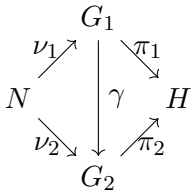
- A (*group*) *extension* of H by N is a short exact sequence of groups

$$1 \to N \xrightarrow{\nu} G \xrightarrow{\pi} H \to 1,$$

i. e. $N \cong \nu(N) \trianglelefteq G$ and $G/\nu(N) \cong \pi(G) = H$. [9]

[9] Note: This terminology is not uniform in the literature.

- We call G a *split* extension, if a homomorphism $\rho\colon H \to G$ with $\pi \circ \rho = \mathrm{id}_H$ exists. In this case, $\rho(H)$ is a complement of $\nu(N)$ in G and $G \cong N \rtimes H$. Conversely, every semidirect product is a split extension.
- Two group extensions G_1, G_2 of H by N are called *equivalent*, if a homomorphism $\gamma\colon G_1 \to G_2$ with $\gamma \circ \nu_1 = \nu_2$ and $\pi_2 \circ \gamma = \pi_1$ exists. That is, the following diagram commutes:



Remark 4.3.

(i) Every group G with normal subgroup N is an extension of $H = G/N$ by N, by choosing the inclusion for ν and the canonical epimorphism for π.

(ii) Let G_1 and G_2 be equivalent extensions via $\gamma\colon G_1 \to G_2$. Let $x \in \mathrm{Ker}(\gamma)$. Because of $\pi_1(x) = \pi_2(\gamma(x)) = 1$, we have $x \in \nu_1(N)$, say $x = \nu_1(y)$. From $\nu_2(y) = \gamma(\nu_1(y)) = \gamma(x) = 1$ it follows that $y = 1 = x$. Therefore γ is injective. For $g \in G_2$ there exists $x \in G_1$ with $\pi_2(\gamma(x)) = \pi_1(x) = \pi_2(g)$. Thus there exists $y \in N$ with $g^{-1}\gamma(x) = \nu_2(y) = \gamma(\nu_1(y))$. This shows $g = \gamma(G_1)$ and γ is surjective. Overall $G_1 \cong G_2$ holds. The equivalence of groups is therefore an equivalence relation.

(iii) Isomorphic extensions, however, do not have to be equivalent: Let

$$G_1 = G_2 = \langle x, y \mid x^4 = y^2 = 1,\ yxy = x^{-1} \rangle \cong D_8,$$

$N = \langle x^2, y \rangle \cong C_2^2$ and $H = \langle xy \rangle$. Let ν_1 be the inclusion, $\nu_2(x^2) = y$ and $\nu_2(y) = x^2$. Let π_1 and π_2 be the canonical epimorphisms. Because of $\mathrm{Z}(G_i) = \langle x^2 \rangle$ there can be no $\gamma \in \mathrm{Aut}(G_i)$ with $\gamma(x^2) = \gamma(\nu_1(x^2)) = \nu_2(x^2) = y$. Nevertheless, it is useful to be able to determine extensions up to equivalence.

(iv) Let G_1 and G_2 be equivalent extensions of H by N with respect to $\gamma\colon G_1 \to G_2$. Suppose G_1 splits with $\rho_1\colon H \to G_1$ and $\pi_1\rho_1 = \mathrm{id}_H$. For $\rho_2 := \gamma\rho\colon H \to G_2$ we have $\pi_2\rho_2 = \pi_2\gamma\rho_1 = \pi_1\rho_1 = \mathrm{id}_H$. Thus G_2 also splits.

(v) Without the methods described below, one can construct extensions in GAP with the `grpconst` package:

```
LoadPackage("grpconst",false);
G:=AlternatingGroup(6);; #the package expects a permutation group
up:=UpwardsExtensions(G,2)[2]; #extensions of G by C_2
List(up,IdGroup);
P:=DihedralGroup(IsPermGroup,32);;
up:=CyclicExtensions(P,2);; #is faster, but does not reduce up to isomorphism
Size(up)=Size(Set(up,IdGroup)); #false, so list is redundant
UpwardsExtensionsNoCentre(G,2); #only works for G = G' and Z(G) = 1
```

Definition 4.4. Let H and N be groups. For $x \in N$, let $c_x \in \mathrm{Inn}(N)$ be the inner automorphism (i. e. $c_x(y) = xyx^{-1}$). A pair of maps

$$\alpha\colon H \to \mathrm{Aut}(N),\ x \mapsto \alpha_x, \qquad \kappa\colon H \times H \to N$$

is called a *parameter system* of H with N if for all $x, y, z \in H$ the following holds:

- $\boxed{\alpha_x \alpha_y = c_{\kappa(x,y)} \alpha_{xy},}$
- $\boxed{\kappa(x,y)\kappa(xy,z) = \alpha_x(\kappa(y,z))\kappa(x,yz).}$

If applicable, α is called an *automorphism system* and κ a *factor system*. A parameter system (or factor system) is called *normalized* if $\kappa(x,1) = \kappa(1,x) = 1$ for all $x \in H$ (in this case $\alpha_1 = \mathrm{id}_N$). Finally, κ is called *trivial* if $\kappa(x,y) = 1$ for all $x, y \in G$ (in this case α is a homomorphism).

Lemma 4.5. *Every extension of H by N determines a (normalized) parameter system.*

Proof. Let $N \xrightarrow{\nu} G \xrightarrow{\pi} H$ be an extension. For $x \in H$ we choose $\tilde{x} \in G$ with $\pi(\tilde{x}) = x$ and $\tilde{1} = 1$. Then $\alpha_x \colon N \to N$, $y \mapsto \nu^{-1}(\tilde{x}\nu(y)\tilde{x}^{-1})$ is an automorphism (note: ν is injective) and $\alpha_1 = \mathrm{id}_N$. For $x, y \in H$ we have $\pi(\widetilde{xy}) = xy = \pi(\tilde{x})\pi(\tilde{y}) = \pi(\tilde{x}\tilde{y})$. Therefore there exists $\kappa(x,y) \in N$ with $\tilde{x}\tilde{y} = \nu(\kappa(x,y))\widetilde{xy}$. Here $\kappa(x,1) = \kappa(1,x) = 1$ holds. For $g \in N$ we have

$$\begin{aligned}(\alpha_x\alpha_y)(g) &= \alpha_x(\nu^{-1}(\tilde{y}\nu(g)\tilde{y}^{-1})) = \nu^{-1}(\tilde{x}\tilde{y}\nu(g)\tilde{y}^{-1}\tilde{x}^{-1}) = \nu^{-1}(\nu(\kappa(x,y))\widetilde{xy}\nu(g)\widetilde{xy}^{-1}\nu(\kappa(x,y)^{-1})) \\ &= \kappa(x,y)\nu^{-1}(\widetilde{xy}\nu(g)\widetilde{xy}^{-1})\kappa(x,y)^{-1} = c_{\kappa(x,y)}\alpha_{xy}(g).\end{aligned}$$

For $x, y, z \in H$ we have

$$\begin{aligned}\nu(\kappa(x,y))\nu(\kappa(xy,z))\widetilde{xyz} &= \nu(\kappa(x,y))\widetilde{xy}\tilde{z} = (\tilde{x}\tilde{y})\tilde{z} = \tilde{x}(\tilde{y}\tilde{z}) = \tilde{x}\nu(\kappa(y,z))\widetilde{yz} \\ &= \nu(\alpha_x(\kappa(y,z)))\tilde{x}\widetilde{yz} = \nu(\alpha_x(\kappa(y,z)))\nu(\kappa(x,yz))\widetilde{xyz}.\end{aligned}$$

Since ν is injective, the claim follows. □

Example 4.6.

(i) If $G = N \rtimes H$, then one can choose $\tilde{x} \in H$ in the above proof. One then obtains the trivial factor system and for $\alpha \colon H \to \mathrm{Aut}(N)$ the conjugation action.

(ii) If $x \in H$ with $x^2 \neq 1$, then one can choose $\widetilde{x^{-1}} = \tilde{x}^{-1}$ in the above proof. One then obtains $\kappa(x, x^{-1}) = 1$.

(iii) Let $G = \langle x, y \rangle = Q_8$, $N = \langle x \rangle \cong C_4$, ν the inclusion and $H = \langle z \rangle \cong C_2$. We can choose $\tilde{z} = y$. One obtains $\kappa(z,z) = \tilde{z}^2(\widetilde{z^2})^{-1} = y^2 \neq 1$.

(iv) Let $n \in \mathbb{N}$ and $H = \langle x \rangle \cong C_n \cong \langle a \rangle = N$. For $i, j \in \mathbb{Z}$ let $\alpha_{x^i} := \mathrm{id}_N$ and $\kappa(x^i, x^j) := a^{ij}$ (well-defined!). Then

$$\kappa(x^i, x^j)\kappa(x^i x^j, x^k) = x^{ij+(i+j)k} = x^{jk+i(j+k)} = \kappa(x^j, x^k)\kappa(x^i, x^j x^k).$$

Therefore (α, κ) is a normalized parameter system.

Lemma 4.7. *Every normalized parameter system of H with N determines an extension.*

Proof. Let (α,κ) be a normalized parameter system of H with N. We consider the set $G = N \times H$ with the operation

$$(a,x)(b,y) := (a\alpha_x(b)\kappa(x,y), xy).$$

Then

$$\begin{aligned}\big((a,x)(b,y)\big)(c,z) &= (a\alpha_x(b)\kappa(x,y),xy)(c,z) = (a\alpha_x(b)\kappa(x,y)\alpha_{xy}(c)\kappa(xy,z),xyz)\\ &= (a\alpha_x(b)\kappa(x,y)\alpha_{xy}(c)\kappa(x,y)^{-1}\kappa(x,y)\kappa(xy,z),xyz)\\ &= (a\alpha_x(b)\alpha_x(\alpha_y(c))\alpha_x(\kappa(y,z))\kappa(x,yz),xyz)\\ &= (a\alpha_x(b\alpha_y(c)\kappa(y,z))\kappa(x,yz),xyz)\\ &= (a,x)(b\alpha_y(c)\kappa(y,z),yz) = (a,x)\big((b,y)(c,z)\big).\end{aligned}$$

The operation is therefore associative. Since κ is normalized, $(1,1)$ is an identity element. Inverse elements are obtained by

$$(\kappa(y^{-1},y)^{-1}\alpha_{y^{-1}}(b)^{-1},y^{-1})(b,y) = (\kappa(y^{-1},y)^{-1}\alpha_{y^{-1}}(b)^{-1}\alpha_{y^{-1}}(b)\kappa(y^{-1},y),y^{-1}y) = (1,1).$$

Thus G is a group. Obviously $\nu\colon N \to G$, $a \mapsto (a,1)$ is a monomorphism and $\pi\colon G \to H$, $(a,x) \mapsto x$ is an epimorphism with $\nu(N) = N \times 1 = \mathrm{Ker}(\pi)$. □

Remark 4.8.

(i) For the trivial factor system, the above proof yields the semidirect product $N \rtimes H$.

(ii) Let (α,κ) be a normalized parameter system with $\alpha_x = \mathrm{id}_N$ for all $x \in H$ (we write $\alpha = 1$ for this). Because of $c_{\kappa(x,y)} = c_{\kappa(x,y)}\alpha_{xy} = \alpha_x\alpha_y = \mathrm{id}_N$, it follows that $\kappa(H \times H) \leq \mathrm{Z}(N)$. Let G be the corresponding extension and $K := \langle (1,x) : x \in H\rangle$. Because of

$$\begin{aligned}(a,x)(1,y)(a,x)^{-1} &= (a\kappa(x,y),xy)(\kappa(x^{-1},x)^{-1}a^{-1},x^{-1})\\ &= (a\kappa(x,y)\kappa(x^{-1},x)^{-1}a^{-1}\kappa(xy,x^{-1}),xyx^{-1}) = (1,x)(1,y)(1,x)^{-1} \in K\end{aligned}$$

$K \trianglelefteq G$ and $N \cap K \leq \kappa(H \times H) \leq \mathrm{Z}(G)$ hold. Thus $G = N * K$ is a central product.

Definition 4.9. Two parameter systems (α,κ), (α',κ') are called *equivalent*, if a map $\varphi\colon H \to N$ exists with

- $\boxed{\alpha'_x = c_{\varphi(x)}\alpha_x,}$
- $\boxed{\kappa'(x,y) = \varphi(x)\alpha_x(\varphi(y))\kappa(x,y)\varphi(xy)^{-1}}$

for all $x,y \in H$. If applicable, we write $(\alpha,\kappa) \sim (\alpha',\kappa')$.

Remark 4.10. The maps $H \to N$ form a group $C^1(H,N)$ w.r.t. $(\varphi\psi)(x) := \varphi(x)\psi(x)$ $(x \in H)$, which is isomorphic to $\times_{h\in H} N$. We show that ${}^{\varphi}(\alpha,\kappa) := (\alpha',\kappa')$ as in Definition 4.9 defines an action of $C^1(H,N)$ on the set of parameter systems. First, it must be shown that (α',κ') is a parameter system:

$$\begin{aligned}\alpha'_x\alpha'_y &= c_{\varphi(x)}\alpha_x c_{\varphi(y)}\alpha_y = c_{\varphi(x)}\alpha_x c_{\varphi(y)}\alpha_x^{-1}\alpha_x\alpha_y = c_{\varphi(x)}c_{\alpha_x(\varphi(y))}c_{\kappa(x,y)}\alpha_{xy} = c_{\kappa'(x,y)}\alpha'_{xy}\\ \kappa'(x,y)\kappa'(xy,z) &= \varphi(x)\alpha_x(\varphi(y))\kappa(x,y)\alpha_{xy}(\varphi(z))\kappa(xy,z)\varphi(xyz)^{-1}\\ &= \varphi(x)\alpha_x(\varphi(y))(c_{\kappa(x,y)}\alpha_{xy})(\varphi(z))\kappa(x,y)\kappa(xy,z)\varphi(xyz)^{-1}\\ &= \varphi(x)\alpha_x(\varphi(y))(\alpha_x\alpha_y)(\varphi(z))\alpha_x(\kappa(y,z))\kappa(x,yz)\varphi(xyz)^{-1}\\ &= \varphi(x)\alpha_x(\varphi(y)\alpha_y(\varphi(z))\kappa(y,z))\kappa(x,yz)\varphi(xyz)^{-1}\\ &= (c_{\varphi(x)}\alpha_x)(\kappa'(y,z))\varphi(x)\alpha_x(\varphi(yz))\kappa(x,yz)\varphi(xyz)^{-1}\\ &= \alpha'_x(\kappa'(y,z))\kappa'(x,yz).\end{aligned}$$

Certainly ${}^1(\alpha,\kappa)=(\alpha,\kappa)$. Let $\varphi,\psi\in C^1(H,N)$ with ${}^\psi(\alpha,\kappa)=(\alpha',\kappa')$. Then it holds that

$$\begin{aligned} c_{\varphi(x)}\alpha'_x &= c_{\varphi(x)}c_{\psi(x)}\alpha_x = c_{\varphi(x)\psi(x)}\alpha_x = c_{(\varphi\psi)(x)}\alpha_x,\\ \varphi(x)\alpha'_x(\varphi(y))\kappa'(x,y)\varphi(xy)^{-1} &= \varphi(x)\psi(x)\alpha_x(\varphi(y))\alpha_x(\psi(y))\kappa(x,y)\psi(xy)^{-1}\varphi(xy)^{-1}\\ &= (\varphi\psi)(x)\alpha_x((\varphi\psi)(y))\kappa(x,y)(\varphi\psi)(xy)^{-1}. \end{aligned}$$

This shows ${}^\varphi({}^\psi(\alpha,\kappa))={}^{\varphi\psi}(\alpha,\kappa)$. The equivalence classes of parameter systems are thus the orbits under $C^1(H,N)$. In particular, $\sim$ is an equivalence relation. The number of equivalence classes could be determined using Burnside's Lemma.

Lemma 4.11. *Every parameter system is equivalent to a normalized parameter system.*

Proof. Let (α,κ) be an arbitrary parameter system and $\zeta:=\kappa(1,1)\in N$. Because of $\alpha_1\alpha_1=c_\zeta\alpha_1$, we have $\alpha_1=c_\zeta$. From

$$\zeta\kappa(1\cdot 1,x)=\alpha_1(\kappa(1,x))\kappa(1,1x)=\zeta\kappa(1,x)\zeta^{-1}\kappa(1,x)$$

it follows that $\kappa(1,x)=\zeta$ for all $x\in H$. With $\kappa(x,1)\kappa(x1,1)=\alpha_x(\zeta)\kappa(x,1\cdot 1)$, we have $\kappa(x,1)=\alpha_x(\zeta)$. Now define $\varphi(1):=\zeta^{-1}$ and $\varphi(y):=1$ for $y\in H\setminus\{1\}$. Then it holds that

$$\begin{aligned} \varphi(x)\alpha_x(\varphi(1))\kappa(x,1)\varphi(x)^{-1} &= \varphi(x)\alpha_x(\zeta)^{-1}\alpha_x(\zeta)\varphi(x)^{-1}=1,\\ \varphi(1)\alpha_1(\varphi(x))\kappa(1,x)\varphi(x)^{-1} &= \zeta^{-1}\zeta\varphi(x)\zeta^{-1}\zeta\varphi(x)^{-1}=1 \end{aligned}$$

for all $x\in H$. In this way, one obtains an equivalent normalized parameter system. □

Lemma 4.12. *Equivalent extensions define equivalent parameter systems.*

Proof. Let $N\xrightarrow{\nu_1}G_1\xrightarrow{\pi_1}H$ and $N\xrightarrow{\nu_2}G_2\xrightarrow{\pi_2}H$ be equivalent extensions via $\gamma\colon G_1\to G_2$. For $x\in H$ let $\tilde{x}\in G_1$ and $\bar{x}\in G_2$ with $\pi_1(\tilde{x})=x=\pi_2(\bar{x})$. As in the proof of Lemma 4.5, one obtains parameter systems (α,κ) and (α',κ') with

$$\begin{aligned} \alpha_x(g) &= \nu_1^{-1}(\tilde{x}\nu_1(g)\tilde{x}^{-1}), & \kappa(x,y) &= \nu_1^{-1}(\tilde{x}\tilde{y}\widetilde{xy}^{-1}),\\ \alpha'_x(g) &= \nu_2^{-1}(\bar{x}\nu_2(g)\bar{x}^{-1}), & \kappa'(x,y) &= \nu_2^{-1}(\bar{x}\bar{y}\overline{xy}^{-1}) \end{aligned}$$

for $x,y\in H$ and $g\in N$. Because of $\pi_2(\gamma(\tilde{x}))=\pi_1(\tilde{x})=x=\pi_2(\bar{x})$, we can define

$$\varphi(x):=\nu_2^{-1}(\bar{x}\gamma(\tilde{x})^{-1})\in N$$

for $x\in H$. Because of $\gamma\nu_1=\nu_2$, it holds that $\varphi(x)=\nu_1^{-1}(\gamma^{-1}(\bar{x})\tilde{x}^{-1})$. It follows that

$$\begin{aligned} (c_{\varphi(x)}\alpha_x)(g) &= \nu_1^{-1}\big(\gamma^{-1}(\bar{x})\tilde{x}^{-1}\big)\nu_1^{-1}\big(\tilde{x}\nu_1(g)\tilde{x}^{-1}\big)\nu_1^{-1}\big(\tilde{x}\gamma^{-1}(\bar{x})^{-1}\big)=\nu_1^{-1}\big(\gamma^{-1}(\bar{x})\nu_1(g)\gamma^{-1}(\bar{x})^{-1}\big)\\ &= \nu_1^{-1}\big(\gamma^{-1}(\bar{x}\nu_2(g)\bar{x}^{-1})\big)=\nu_2^{-1}\big(\bar{x}\nu_2(g)\bar{x}^{-1}\big)=\alpha'_x(g) \end{aligned}$$

and

$$\begin{aligned} \varphi(x)\alpha_x(\varphi(y))\kappa(x,y)\varphi(xy)^{-1} &= \nu_1^{-1}\big(\gamma^{-1}(\bar{x})\tilde{x}^{-1}\big)\nu_1^{-1}\big(\tilde{x}\nu_1(\varphi(y))\tilde{x}^{-1}\big)\nu_1^{-1}\big(\tilde{x}\tilde{y}\widetilde{xy}^{-1}\big)\nu_1^{-1}\big(\widetilde{xy}\gamma^{-1}(\overline{xy})^{-1}\big)\\ &= \nu_1^{-1}\big(\gamma^{-1}(\bar{x})\nu_1(\varphi(y))\tilde{y}\gamma^{-1}(\overline{xy})^{-1}\big)\\ &= \nu_1^{-1}\big(\gamma^{-1}(\bar{x})\gamma^{-1}(\bar{y})\gamma^{-1}(\overline{xy})^{-1}\big)=\nu_2^{-1}\big(\bar{x}\bar{y}\overline{xy}^{-1}\big)=\kappa'(x,y). \end{aligned}$$

Therefore, (α,κ) and (α',κ') are equivalent. □

Lemma 4.13. *Equivalent (normalized) parameter systems define equivalent extensions.*

Proof. Let (α, κ) and (α', κ') be equivalent parameter systems via $\varphi\colon H \to N$. According to Lemma 4.11, we can assume that both parameter systems are normalized. Then it holds that

$$1 = \kappa'(1,x) = \varphi(1)\alpha_1(\varphi(x))\kappa(1,x)\varphi(x)^{-1} = \varphi(1).$$

We construct $(G_1, \cdot)$ and $(G_2, *)$ as in the proof of Lemma 4.7. Since G_1 and G_2 are equal as sets, we can define $\gamma\colon G_1 \to G_2$, $(a,x) \mapsto (a\varphi(x)^{-1}, x)$. For $(a,x),(b,y) \in G_1$ it holds that

$$\begin{aligned}\gamma(a,x) * \gamma(b,y) &= (a\varphi(x)^{-1}, x) * (b\varphi(y)^{-1}, y) = (a\varphi(x)^{-1}\alpha'_x(b\varphi(y)^{-1})\kappa'(x,y), xy)\\ &= (a\alpha_x(b\varphi(y)^{-1})\alpha_x(\varphi(y))\kappa(x,y)\varphi(xy)^{-1}, xy) = (a\alpha_x(b)\kappa(x,y)\varphi(xy)^{-1}, xy)\\ &= \gamma(a\alpha_x(b)\kappa(x,y), xy) = \gamma((a,x)\cdot(b,y)),\end{aligned}$$

i. e. γ is a homomorphism. Furthermore, it holds that

$$\begin{aligned}(\gamma\nu_1)(a) &= \gamma(a,1) = (a\varphi(1)^{-1}, 1) = (a,1) = \nu_2(a),\\ (\pi_2\gamma)(a,x) &= \pi_2(a\varphi(x)^{-1}, x) = x = \pi_1(a,x).\end{aligned}$$

Thus G_1 and G_2 are equivalent. □

Theorem 4.14 (Schreier)**.** *There is a bijection between the set of equivalence classes of extensions of H by N and the set of equivalence classes of (normalized) parameter systems of H with N.*

Proof. According to Lemma 4.12 and Lemma 4.13, there are well-defined maps between the sets of equivalence classes. We show that they are inverse to each other.

Let the extension $N \xrightarrow{\nu_1} G_1 \xrightarrow{\pi_1} H$ define the normalized parameter system (α, κ) with the elements $\tilde{x} \in G_1$ as in Lemma 4.5. From (α, κ) we construct the extension $N \xrightarrow{\nu_2} G_2 \xrightarrow{\pi_2} H$ as in Lemma 4.7. Every element in G_1 can be uniquely written in the form $\nu_1(a)\tilde{x}$ with $a \in N$ and $x \in H$. We define $\gamma\colon G_1 \to G_2$, $\nu_1(a)\tilde{x} \mapsto (a,x)$. For $\nu_1(a)\tilde{x}, \nu_1(b)\tilde{y} \in G_1$ we have

$$\begin{aligned}\gamma\big(\nu_1(a)\tilde{x}\cdot\nu_1(b)\tilde{y}\big) &= \gamma\big(\nu_1(a)\tilde{x}\nu_1(b)\tilde{x}^{-1}\cdot\tilde{x}\tilde{y}\big) = \gamma\big(\nu_1(a)\nu_1(\alpha_x(b))\cdot\nu_1(\kappa(x,y))\widetilde{xy}\big)\\ &= (a\alpha_x(b)\kappa(x,y), xy) = (a,x) * (b,y) = \gamma(\nu_1(a)\tilde{x}) * \gamma(\nu_2(b), y),\end{aligned}$$

i. e. γ is a homomorphism with

$$\begin{aligned}(\gamma\nu_1)(a) &= \gamma(\nu_1(a)\tilde{1}) = (a,1) = \nu_2(a),\\ (\pi_2\gamma)(\nu_1(a)\tilde{x}) &= \pi_2(a,x) = x = \pi_1(\tilde{x}) = \pi_1(\nu_1(a)\tilde{x}).\end{aligned}$$

Thus G_1 and G_2 are equivalent.

Conversely, let first (α, κ) be given normalized. We construct the extension G and from it the parameter system (α', κ') by means of the elements $\tilde{x} = (1,x) \in G$ for $x \in H$. First, we have

$$\kappa(x,x^{-1}) = \kappa(x,x^{-1})\kappa(xx^{-1},x) = \alpha_x(\kappa(x^{-1},x))\kappa(x,x^{-1}x) = \alpha_x(\kappa(x^{-1},x)) \tag{4.1}$$

for $x \in H$. From this it follows that

$$\begin{aligned}
\alpha'_x(g) &= \nu^{-1}(\tilde{x}\nu(g)\tilde{x}^{-1}) = \nu^{-1}\big((1,x)(g,1)(1,x)^{-1}\big) = \nu^{-1}\big((\alpha_x(g),x)(\kappa(x^{-1},x)^{-1},x^{-1})\big)\\
&= \nu^{-1}\big(\alpha_x(g)\alpha_x(\kappa(x^{-1},x)^{-1})\kappa(x,x^{-1}),1\big) = \alpha_x(g),\\
\kappa'(x,y) &= \nu^{-1}\big(\tilde{x}\tilde{y}\widetilde{xy}^{-1}\big) = \nu^{-1}\big((1,x)(1,y)(\kappa((xy)^{-1},xy)^{-1},(xy)^{-1})\big)\\
&= \nu^{-1}\big((\kappa(x,y),xy)(\kappa((xy)^{-1},xy)^{-1},(xy)^{-1})\big)\\
&= \nu^{-1}\big(\kappa(x,y)\alpha_{xy}(\kappa((xy)^{-1},xy)^{-1})\kappa(xy,(xy)^{-1}),1\big)\\
&= \nu^{-1}(\kappa(x,y),1) = \kappa(x,y)
\end{aligned}$$

for $x,y \in H$ and $g \in N$. This shows $(\alpha,\kappa) = (\alpha',\kappa')$. □

Remark 4.15. According to Remark 4.3, the split extensions correspond up to equivalence exactly to the parameter systems with trivial factor system.

Theorem 4.16. *Let $H = \langle x\rangle \cong C_n$ and $\beta \in \mathrm{Aut}(N)$. A parameter system (α,κ) with $\alpha_x = \beta$ exists if and only if there exists an $a \in N$ with $\beta(a) = a$ and $\beta^n = c_a$.*

Proof. Let (α,κ) be a parameter system with $\alpha_x = \beta$. In the case $n = 1$, the claim holds with $a := \kappa(1,1)$, because

$$\begin{aligned}
\alpha_1\alpha_1 &= c_{\kappa(1,1)}\alpha_1,\\
\kappa(1,1)\kappa(1,1) &= \alpha_1(\kappa(1,1))\kappa(1,1).
\end{aligned}$$

For $n \geq 2$, we may pass to a normalized parameter system without changing α_x (see proof of Lemma 4.11). We construct the corresponding extension $G = N \times H$. For $\tilde{x} := (1,x) \in G$ and $b \in N$, it holds that

$$\tilde{x}\nu(b)\tilde{x}^{-1} = (1,x)(b,1)(1,x)^{-1} = (\alpha_x(b),x)(\kappa(x^{-1},x)^{-1},x^{-1}) \overset{(4.1)}{=} (\alpha_x(b),1) = \nu(\beta(b)).$$

Thus $\beta = \nu^{-1}c_{\tilde{x}}\nu$. Let $(a,1) := \tilde{x}^n$. Then $\beta^n = \nu^{-1}c_{(a,1)}\nu = c_a$ and

$$\beta(a) = \nu^{-1}(\tilde{x}\nu(a)\tilde{x}^{-1}) = \nu^{-1}(\tilde{x}\tilde{x}^n\tilde{x}^{-1}) = \nu^{-1}(a,1) = a.$$

Conversely, let $\beta^n = c_a$ and $\beta(a) = a$ for some $a \in N$. We define $\alpha_{x^i} := \beta^i$ and

$$\kappa(x^i,x^j) := \begin{cases} 1 & \text{if } i+j < n,\\ a & \text{if } i+j \geq n \end{cases}$$

for $0 \leq i,j \leq n-1$. Then it holds that

$$\alpha_{x^i}\alpha_{x^j} = \beta^{i+j} = \begin{cases} \alpha_{x^{i+j}} = c_{\kappa(x^i,x^j)}\alpha_{x^ix^j} & \text{if } i+j < n,\\ c_a\beta^{i+j-n} = c_{\kappa(x^i,x^j)}\alpha_{x^ix^j} & \text{if } i+j \geq n \end{cases}$$

and

$$\kappa(x^i,x^j)\kappa(x^{i+j},x^k) = \begin{cases} 1 & \text{if } i+j+k < n,\\ a & \text{if } n \leq i+j+k < 2n,\\ a^2 & \text{if } i+j+k \geq 2n. \end{cases}$$

Because of $\alpha_{x^i}(a) = \beta^i(a) = a$, it holds that $\kappa(x^i,x^j)\kappa(x^{i+j},x^k) = \alpha_{x^i}(\kappa(x^j,x^k))\kappa(x^i,x^{j+k})$ in all cases. Thus (α,κ) is a parameter system. □

Example 4.17.

(i) If β in the situation of Theorem 4.16 is an inner automorphism, say $\beta = c_b$ with $b \in N$, then the requirements hold with $a = b^n$. Thus there is always an extension such that β is induced by x.

(ii) If N is abelian, then the conditions reduce to $\beta^n = 1$. For non-abelian groups, this is not sufficient (Exercise 17).

Remark 4.18. Let N be abelian. Then the automorphism systems are precisely the homomorphisms $\alpha\colon H \to \mathrm{Aut}(N)$. The corresponding factor systems form a subgroup $F_\alpha \leq C^2(H,N) := C^1(H \times H, N)$ (for a fixed α). Furthermore, $(\alpha,\kappa) \sim (\alpha',\kappa')$ implies $\alpha = \alpha'$. One can therefore investigate the equivalence classes within F_α.

Theorem 4.19. *Let N be abelian and $\alpha\colon H \to \mathrm{Aut}(N)$ a homomorphism. Then*

$$P_\alpha := \{\kappa \in F_\alpha : (\alpha,\kappa) \sim (\alpha,1)\} \leq F_\alpha.$$

The equivalence classes of parameter systems with automorphism system α correspond to the elements of $\overline{F_\alpha} := F_\alpha/P_\alpha$.

Proof. For $\kappa,\kappa' \in P_\alpha$ there exist $\varphi,\psi \in C^1(H,N)$ with $\kappa(x,y) = \varphi(x)\alpha_x(\varphi(y))\varphi(xy)^{-1}$ and $\kappa'(x,y) = \psi(x)\alpha_x(\psi(y))\psi(xy)^{-1}$ for $x,y \in H$. Since N is abelian, it follows that

$$(\kappa\kappa')(x,y) = (\varphi\psi)(x)\alpha_x((\varphi\psi)(y))(\varphi\psi)(xy)^{-1}$$

and $\kappa\kappa' \in P_\alpha$. One easily obtains $P_\alpha \leq F_\alpha$. Since $C^2(H,N) \cong \times_{x\in H\times H} N$ is abelian, $P_\alpha \trianglelefteq F_\alpha$ holds and $\overline{F_\alpha}$ is well-defined. Furthermore,

$$\begin{aligned}(\alpha,\kappa) \sim (\alpha,\kappa') &\iff \exists \varphi \in C^1(H,N)\, \forall x,y \in H : \kappa'(x,y) = \varphi(x)\alpha_x(\varphi(y))\kappa(x,y)\varphi(xy)^{-1} \\ &\iff \exists \varphi \in C^1(H,N) : \kappa^{-1}\kappa' \in P_\alpha \iff \kappa P_\alpha = \kappa' P_\alpha.\end{aligned}$$

□

Remark 4.20. Extensions can be calculated on the computer particularly efficiently for *polycyclic* groups H. This means that H possesses a subnormal series with cyclic factors. For finite groups, polycyclic is equivalent to solvable. We furthermore assume that N is an elementary abelian p-group. Through the homomorphism $\alpha\colon H \to \mathrm{Aut}(N)$, N then becomes an $\mathbb{F}_pH$-module.

```
H:=SymmetricGroup(4);; #permutation group
H:=Image(IsomorphismPcGroup(H));; #isomorphic polycyclic group
IrrM:=IrreducibleModules(H,GF(2),2); #simple F_2H-modules of dimension ≤ 2
N:=IrrM[2][2];; #a 2-dimensional simple F_2H-module, so N ≅ C_2^2
ext:=Extensions(H,N); #two extensions of order 96
List(ext,IdGroup); #number in the small groups library
IdGroup(SplitExtension(H,N)); #splitting extension
Fa:=VectorSpace(GF(2),TwoCocycles(H,N));; #= F_α
Pa:=Subspace(Fa,TwoCoboundaries(H,N));; #= P_α
t:=Difference(Fa,Pa)[1];; #non-trivial factor system
G:=Extension(H,N,t);; #this should be the non-splitting extension
IdGroup(G);
TwoCohomology(H,N); #more info about F_α
```

For non-polycyclic groups one can include the `cohomolo` package (so far only under Linux). It expects as input a permutation group and an isomorphic finitely presented group.

```
LoadPackage("cohomolo",false); #false suppresses banner
H:=GL(5,2);;
gen:=GeneratorsOfGroup(H);;
Hp:=Image(IsomorphismPermGroup(H));;
Hf:=Image(IsomorphismFpGroupByGenerators(H,gen));;
chr:=CHR(Hp,2,Hf,gen); #here N = F_2^5 and α: H → Aut(N) is the natural isomorphism
SecondCohomologyDimension(chr); #=1 means F_α ≅ F_2, i.e. there are two extensions
G:=NonsplitExtension(chr); #this is the DEMPWOLFF group
Size(G); #takes forever
```

Theorem 4.21. *Let N be abelian, $K \leq H$ finite and $\alpha\colon H \to \mathrm{Aut}(N)$ a homomorphism. Let $\alpha_K\colon K \to \mathrm{Aut}(N)$ be the restriction of α. Then the map*

$$\overline{F_\alpha} \to \overline{F_{\alpha_K}}, \qquad \kappa P_\alpha \mapsto \kappa_{K\times K} P_{\alpha_K}$$

is a homomorphism. From $\kappa_{K\times K} \in P_{\alpha_K}$ it follows that $\kappa^{|H:K|} \in P_\alpha$. In particular, $\exp(\overline{F_\alpha})$ is a divisor of $|H|$.

Proof. For $\kappa \in F_\alpha$ it is obvious that $\kappa_{K\times K} \in F_{\alpha_K}$ and the map $F_\alpha \to F_{\alpha_K}$, $\kappa \mapsto \kappa_{K\times K}$ is a homomorphism. From $\kappa \equiv \kappa' \pmod{P_\alpha}$ it follows that $\kappa_{K\times K} \equiv \kappa'_{K\times K} \pmod{F_{\alpha_K}}$. Thus $\overline{F_\alpha} \to \overline{F_{\alpha_K}}$ is a well-defined homomorphism.

Now let $\kappa_{K\times K} \in P_{\alpha_K}$. Let G be an extension of H by N w.r.t. (α,κ). Then there exists a homomorphism $\rho\colon K \to G$ with $\pi\rho = \mathrm{id}_K$. Let R be a transversal for H/K with $1 \in R$. For $r \in R$ let $\tilde{r} \in G$ with $\pi(\tilde{r}) = r$ and $\tilde{1} = 1$. For $x \in H$ let $r_x \in R$ with $r_x^{-1}x \in K$. We define $\tilde{x} := \widetilde{r_x}\rho(r_x^{-1}x)$. Then $\pi(\tilde{x}) = r_x r_x^{-1} x = x$ and

$$\kappa'(x,y) := \nu^{-1}(\tilde{x}\tilde{y}\widetilde{xy}^{-1})$$

defines a normalized factor system equivalent to (α,κ). For $x,y \in H$ we have $r_{xy}K = xyK = xr_yK = r_{xr_y}K$ and $r_{xy} = r_{xr_y}$. It follows

$$\begin{aligned}\kappa'(x,y) &= \nu^{-1}\big(\tilde{x}\widetilde{r_y}\rho(r_y^{-1}y)\widetilde{xy}^{-1}\big) = \nu^{-1}\big(\tilde{x}\widetilde{r_y}\widetilde{xr_y}^{-1}\widetilde{xr_y}\rho(r_y^{-1}y)\widetilde{xy}^{-1}\big)\\ &= \kappa'(x,r_y)\nu^{-1}\big(\widetilde{xr_y}\rho(r_y^{-1}y)\widetilde{xy}^{-1}\big) = \kappa'(x,r_y)\nu^{-1}\big(\widetilde{r_{xy}}\rho(r_{xy}^{-1}xr_y)\rho(r_y^{-1}y)\widetilde{xy}^{-1}\big) = \kappa'(x,r_y).\end{aligned}$$

For $r \in R$ it follows

$$\kappa'(x,y) = \alpha_x(\kappa'(y,r))\kappa'(x,yr)\kappa'(xy,r)^{-1} = \alpha_x(\kappa'(y,r))\kappa'(x,r_{yr})\kappa'(xy,r)^{-1}.$$

As r runs through R, so does r_{yr}. Let $\varphi(x) := \prod_{r\in R}\kappa'(x,r)$ for $x \in H$. Then

$$\kappa'(x,y)^{|H:K|} = \prod_{r\in R}\alpha_x(\kappa'(y,r))\kappa'(x,r_{yr})\kappa'(xy,r)^{-1} = \alpha_x(\varphi(y))\varphi(x)\varphi(xy)^{-1}.$$

This shows $(\kappa')^{|H:K|} \in P_\alpha$. Because $(\alpha,\kappa) \sim (\alpha,\kappa')$, we also have $\kappa^{|H:K|} \in P_\alpha$ by Theorem 4.19.

The second statement follows by choosing $K = 1$. □

Remark 4.22. The following two theorems were added later to the group theory notes with alternative proofs.

Theorem 4.23 (GASCHÜTZ)**.** *Let G be a finite group with an abelian normal subgroup N. Let $N \leq H \leq G$ with $\gcd(|N|, |G:H|) = 1$, such that N has a complement in H. Then N has a complement in G.*

Proof. As is well known, G is an extension of G/N by N. Let (α, κ) be the corresponding parameter system. By assumption, the extension of $K := H/N$ by N splits. Therefore, $\kappa_{K\times K} \in P_{\alpha_K}$. From Theorem 4.21 it follows that $\kappa^{|G:H|} \in P_\alpha$. Because of $F_\alpha \leq C^2(G/N, N)$, we also have $\kappa^{|N|} \in P_\alpha$. From $\gcd(|G : H|, |N|) = 1$ it follows that $\kappa \in P_\alpha$, i. e. G splits. □

Theorem 4.24 (Gaschütz)**.** *Let G be a finite group with an abelian normal subgroup N. N has a complement in G if and only if N has a complement in P for every Sylow group P/N of G/N.*

Proof. If K is a complement of N in G, then $K \cap P$ is a complement of N in P for $N \leq P \leq G$, since $N(K \cap P) = NK \cap P = P$ and $N \cap (K \cap P) \leq N \cap K = 1$.

Conversely, let us now assume that N has a complement in P for all Sylow groups P/N of G/N. Let (α, κ) be the parameter system of the extension G. According to Theorem 4.21, $\kappa^{|G:P|} \in P_\alpha$ holds. This shows $\kappa^{|G/N|_{p'}} \in P_\alpha$ for all prime divisors p of $|G/N|$. Since the numbers $|G/N|_{p'}$ are coprime (where p runs over all prime divisors of $|G|$), it follows that $\kappa \in P_\alpha$, i. e. G splits. □

Definition 4.25. Now let N be arbitrary and (α, κ) be a parameter system.

- The map $\omega\colon H \to \mathrm{Out}(N)$, $x \mapsto \alpha_x\mathrm{Inn}(N)$ is a homomorphism, which is called the *pairing* of (α, κ). Let $\mathrm{Par}(\omega)$ be the set of parameter systems with pairing ω. Equivalent parameter systems define the same pairing. Let $\overline{\mathrm{Par}(\omega)}$ be the corresponding set of equivalence classes.
- Let $Z := \mathrm{Z}(N)$. Every homomorphism $\omega\colon H \to \mathrm{Out}(N)$, $x \mapsto \omega_x\mathrm{Inn}(N)$ defines a well-defined homomorphism $\omega_Z\colon H \to \mathrm{Aut}(Z)$, $x \mapsto (\omega_x)_{|Z}$.

Remark 4.26. A corresponding parameter system does not necessarily exist for every homomorphism $\omega\colon H \to \mathrm{Out}(N)$ (Exercise 17).

Theorem 4.27. *Let $\omega\colon H \to \mathrm{Out}(N)$ be a homomorphism with $\mathrm{Par}(\omega) \neq \varnothing$. Let $Z := \mathrm{Z}(N)$ and $\beta := \omega_Z$. Then $\overline{F_\beta}$ acts regularly on $\overline{\mathrm{Par}(\omega)}$. In particular, $\overline{\mathrm{Par}(\omega)}$ and $\overline{F_\beta}$ have the same cardinality.*

Proof. Let $\lambda \in F_\beta$ and $(\alpha, \kappa) \in \mathrm{Par}(\omega)$. Because of $\lambda(H \times H) \leq Z$, it holds that

$$\begin{aligned}\alpha_x\alpha_y &= c_{\kappa(x,y)}\alpha_{xy} = c_{(\lambda\kappa)(x,y)}\alpha_{xy},\\ (\lambda\kappa)(x,y)(\lambda\kappa)(xy,z) &= \alpha_x(\kappa(y,z))\beta_x(\lambda(y,z))(\lambda\kappa)(x,yz) = \alpha_x((\lambda\kappa)(y,z))(\lambda\kappa)(x,yz)\end{aligned}$$

for $x, y, z \in H$. Therefore $(\alpha, \lambda\kappa) \in \mathrm{Par}(\omega)$. One easily sees that ${}^\lambda(\alpha, \kappa) := (\alpha, \lambda\kappa)$ defines an action of F_β on $\mathrm{Par}(\omega)$. Let $(\alpha', \kappa') \sim (\alpha, \kappa)$. Then there exists $\varphi \in C^1(H, N)$ with

$$\begin{aligned}\alpha'_x &= c_{\varphi(x)}\alpha_x,\\ (\lambda\kappa')(x,y) &= \varphi(x)\alpha_x(\varphi(y))(\lambda\kappa)(x,y)\varphi(xy)^{-1}.\end{aligned}$$

Thus ${}^\lambda(\alpha, \kappa) \sim {}^\lambda(\alpha', \kappa')$ and F_β acts on $\overline{\mathrm{Par}(\omega)}$. Let $\lambda \in P_\beta$, i.e., $\lambda(x, y) = \delta(x)\beta_x(\delta(y))\delta(xy)^{-1}$ for some $\delta \in C^1(H, Z)$. Then

$$(\lambda\kappa)(x,y) = \delta(x)\alpha_x(\delta(y))\kappa(x,y)\delta(xy)^{-1},$$

i.e., ${}^\lambda(\alpha, \kappa) \sim (\alpha, \kappa)$. Thus P_β acts trivially on $\overline{\mathrm{Par}(\omega)}$ and one obtains a well-defined action of $\overline{F_\beta}$ on $\overline{\mathrm{Par}(\omega)}$. Let $\lambda \in F_\beta$ with ${}^\lambda(\alpha, \kappa) \sim (\alpha, \kappa)$. Then there exists $\varphi \in C^1(H, N)$ with

$$\begin{aligned}\alpha_x &= c_{\varphi(x)}\alpha_x,\\ (\lambda\kappa)(x,y) &= \varphi(x)\alpha_x(\varphi(y))\kappa(x,y)\varphi(xy)^{-1}\end{aligned}$$

for $x, y \in H$. It follows that $\varphi(H) \leq Z$ and $\lambda(x,y) = \varphi(x)\beta_x(\varphi(y))\varphi(xy)^{-1}$. This shows $\lambda \in P_\beta$. Therefore $\overline{F_\beta}$ acts fixed-point-freely on $\overline{\mathrm{Par}(\omega)}$. For transitivity, let $(\alpha,\kappa),(\alpha',\kappa') \in \mathrm{Par}(\omega)$. Because of $\alpha_x\mathrm{Inn}(N) = \omega(x) = \alpha'_x\mathrm{Inn}(N)$ for $x \in H$, there exists $\varphi \in C^1(H,N)$ with $\alpha_x = c_{\varphi(x)}\alpha'_x$ for $x \in H$. We can thus replace (α',κ') by an equivalent parameter system of the form (α,κ'). Now it holds that

$$c_{\kappa(x,y)}\alpha_{xy} = \alpha_x\alpha_y = c_{\kappa'(x,y)}\alpha_{xy}$$

and $\lambda := \kappa^{-1}\kappa' \colon H \times H \to Z$. Because of

$$\begin{aligned}\lambda(x,y)\lambda(xy,z) &= \kappa(xy,z)^{-1}\lambda(x,y)\kappa'(xy,z) = \kappa(xy,z)^{-1}\kappa(x,y)^{-1}\kappa'(x,y)\kappa'(xy,z)\\ &= \big(\alpha_x(\kappa(y,z))\kappa(x,yz)\big)^{-1}\alpha_x(\kappa'(y,z))\kappa'(x,yz)\\ &= \kappa(x,yz)^{-1}\alpha_x\big(\kappa(y,z)^{-1}\kappa'(y,z)\big)\kappa'(x,yz) = \kappa(x,yz)^{-1}\beta_x(\lambda(y,z))\kappa'(x,yz)\\ &= \beta_x(\lambda(y,z))\kappa(x,yz)^{-1}\kappa'(x,yz) = \beta_x(\lambda(y,z))\lambda(x,yz)\end{aligned}$$

for $x, y \in H$, it holds that $\lambda \in F_\beta$ and ${}^\lambda(\alpha,\kappa) = (\alpha,\kappa\lambda) = (\alpha,\kappa')$. Thus $\overline{F_\beta}$ acts regularly on $\overline{\mathrm{Par}(\omega)}$. □

Corollay 4.28. *Let* $\mathrm{Z}(N) = 1$. *Then:*

(i) For every homomorphism $\omega \colon H \to \mathrm{Out}(N)$, *it holds that* $|\overline{\mathrm{Par}(\omega)}| = 1$.

(ii) If $\mathrm{Inn}(N) \cong N$ *possesses a complement in* $\mathrm{Aut}(N)$, *then every extension with* N *splits.*

Proof.

(i) Let $\omega(x) = \alpha_x\mathrm{Inn}(N)$ for $x \in H$. Because of $\alpha_{xy}\mathrm{Inn}(N) = \omega(xy) = \omega(x)\omega(y) = \alpha_x\alpha_y\mathrm{Inn}(N)$, there exist $\kappa(x,y) \in N$ with $\alpha_x\alpha_y = c_{\kappa(x,y)}\alpha_{xy}$ for all $x, y \in H$. Here, it holds that

$$\begin{aligned}c_{\kappa(x,y)\kappa(xy,z)}\alpha_{xyz} &= c_{\kappa(x,y)}\alpha_{xy}\alpha_z = \alpha_x\alpha_y\alpha_z = \alpha_x c_{\kappa(y,z)}\alpha_{yz}\\ &= c_{\alpha_x(\kappa(y,z))}\alpha_x\alpha_{yz} = c_{\alpha_x(\kappa(y,z))\kappa(x,yz)}\alpha_{xyz}.\end{aligned}$$

From $\mathrm{Inn}(N) \cong N/\mathrm{Z}(N) \cong N$, it follows that $\kappa(x,y)\kappa(xy,z) = \alpha_x(\kappa(y,z))\kappa(x,yz)$. This shows $(\alpha,\kappa) \in \mathrm{Par}(\omega) \neq \varnothing$ and $|\overline{\mathrm{Par}(\omega)}| = 1$ according to Theorem 4.27.

(ii) By assumption, there exists a homomorphism $\tau \colon \mathrm{Out}(N) \to \mathrm{Aut}(N)$ with $\tau(\gamma)\mathrm{Inn}(N) = \gamma$ for all $\gamma \in \mathrm{Out}(N)$. Every homomorphism $\omega \colon H \to \mathrm{Out}(N)$ can thus be lifted to a homomorphism $\alpha := \tau\omega \colon H \to \mathrm{Aut}(N)$ (i. e. $\omega(x) = \alpha_x\mathrm{Inn}(N)$ for all $x \in H$). According to (i), the splitting extension with respect to $(\alpha, 1)$ is the only extension with pairing ω. □

Example 4.29. If N is *complete*, i. e. $\mathrm{Z}(N) = 1 = \mathrm{Out}(N)$, then there are only the extensions $N \times H$. This was already shown in GT-exercise 26.

Remark 4.30. The number of extensions with a non-abelian group N for a pairing $\omega \colon K \to \mathrm{Out}(N)$ can be determined with the `hap` package:

```
LoadPackage("hap",false); #loads further packages
N:=QuaternionGroup(8);;
H:=AutomorphismGroup(N);;
NH:=SemidirectProduct(H,N);;
omega:=GOuterGroup(NH,Image(Embedding(NH,2))); #defines pairing H → Out(N)
beta:=Center(omega); #corresponding factor system H → Aut(Z(N))
A:=ActingGroup(beta);; #preimage of β
R:=ResolutionFiniteGroup(A,3);; #three terms of a resolution
C:=HomToGModule(R,alpha);; #corresponding chain complex
Cohomology(C,2); #orders of the cyclic factors of F_β
```

Theorem 4.31 (Johnson-Zassenhaus)**.** *Two extensions $N \xrightarrow{\nu_i} G_i \xrightarrow{\pi_i} H$ $(i = 1, 2)$ are equivalent if and only if for every Sylow group P of H, the extensions $\pi_1^{-1}(P)$ and $\pi_2^{-1}(P)$ of P with N are equivalent.*

Proof. Let G_1 and G_2 be equivalent via $\gamma\colon G_1 \to G_2$. For a prime p and $P \in \mathrm{Syl}_p(H)$ let $P_i := \pi_i^{-1}(P) \leq G_i$ and $\delta := \gamma_{P_1}$. Then

$$\delta(P_1) = \gamma(\pi_1^{-1}(P)) = \gamma\big(\gamma^{-1}(\pi_2^{-1}(P))\big) = P_2,$$

$\delta\nu_1 = \nu_2$ and $(\pi_2)_{|P_2}\delta = (\pi_1)_{|P_1}$. Thus P_1 and P_2 are equivalent.

Now assume that P_1 and P_2 are equivalent for all Sylow subgroups P of H. Let (α, κ) and (α', κ') be the parameter systems of G_1 and G_2, respectively. Then $(\alpha_P, \kappa_{P\times P})$ and $(\alpha'_P, \kappa'_{P\times P})$ are equivalent and it follows that $\alpha_x\mathrm{Inn}(N) = \alpha'_x\mathrm{Inn}(N)$ for all $x \in P$. Since H is generated by its Sylow subgroups, (α, κ) and (α', κ') have the same pairing ω. Let $Z := \mathrm{Z}(N)$ and $\beta := \omega_Z$. By Theorem 4.27 there exists $\lambda \in F_\beta$ with $(\alpha', \kappa') \sim (\alpha, \lambda\kappa)$. By assumption, $\lambda_P \in P_{\beta_P}$ for every Sylow subgroup P of H, i.e., the extension of P by Z splits. By Theorem 4.24, the extension of H by Z now also splits, i.e., $\lambda \in P_\beta$. This shows $(\alpha', \kappa') \sim (\alpha, \lambda\kappa) \sim (\alpha, \kappa)$. □

Remark 4.32. Theorem 4.31 is not suitable for proving Gaschütz's Theorem 4.24 for non-abelian N, because for every homomorphism $\omega\colon H \to \mathrm{Out}(N)$ there does not necessarily exist a splitting extension with which one could compare (Exercise 17). Even if such a splitting extension of H by N exists, the restricted extension of H by $\mathrm{Z}(N)$ does not necessarily have to split. For example, $G = D_8 * C_4$ is a splitting extension of $H = C_2$ by $N = D_8$, but the restricted extension $\mathrm{C}_G(N) = C_4$ of H by $\mathrm{Z}(N) \cong C_2$ does not split. A similar example shows that Theorem 4.24 is in general false for non-abelian groups N (Exercise 15).

5 Central Extensions

Definition 5.1. If N is abelian and $\alpha\colon H \to \mathrm{Aut}(N)$ is trivial, then

$$\begin{aligned}
Z^2(H, N) &:= F_\alpha = \big\{\kappa \in C^2(H, N) : \kappa(x, y)\kappa(xy, z) = \kappa(y, z)\kappa(x, yz)\big\},\\
B^2(H, N) &:= P_\alpha = \big\{\kappa \in Z^2(H, N) : \exists\varphi \in C^1(H, N) : \kappa(x, y) = \varphi(x)\varphi(y)\varphi(xy)^{-1}\big\},\\
H^2(H, N) &:= \overline{F_\alpha} = F_\alpha/P_\alpha.
\end{aligned}$$

The elements of $Z^2(H, N)$ are called (2-)*cocycles*. For $\varphi \in C^1(H, N)$ let $\partial\varphi \in B^2(H, N)$ with $\partial\varphi(x, y) := \varphi(x)\varphi(y)\varphi(xy)^{-1}$ for $x, y \in H$. One calls $H^2(H, N)$ the second *cohomology group* of H with values in N. If G is a corresponding extension, then $\nu(N) \leq \mathrm{Z}(G)$ (cf. Remark 4.8). One therefore speaks of *central* extensions.

Remark 5.2.

(i) In the following, let A always be an abelian group.

(ii) Let H be abelian and G the central extension on $A \times H$ by means of $\kappa \in Z^2(H, A)$, d. h.

$$(a, x)(b, y) = (ab\kappa(x, y), xy)$$

for all $(a, x), (b, y) \in G$. G is abelian if and only if κ is symmetric, d. h. $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in H$. Since H is abelian, the cocycles in $B^2(H, A)$ are symmetric.

Definition 5.3. For abelian groups H let

$$Z_s^2(H,A) := \left\{\kappa \in Z^2(H,A) : \forall x,y \in H : \kappa(x,y) = \kappa(y,x)\right\}$$

and $H_s^2(H,A) := Z_s^2(H,A)/B^2(H,A)$.

Theorem 5.4. *For $n \in \mathbb{N}$ it holds that $H^2(C_n,A) = H_s^2(C_n,A) \cong A/\langle a^n : a \in A\rangle$.*

Proof. Let G be a central extension of $H = \langle x\rangle \cong C_n$ by A. Because of $A \leq \mathrm{Z}(G)$, $G/\mathrm{Z}(G)$ is cyclic and G is abelian. This shows $H^2(H,A) = H_s^2(H,A)$. For $a \in A$, we have constructed in the proof of Theorem 4.16 a cocycle $\kappa_a \in Z^2(H,A)$ with

$$\kappa_a(x^i,x^j) = \begin{cases} 1 & \text{if } i+j < n, \\ a & \text{if } i+j \geq n \end{cases}$$

. The map $F\colon A \to H^2(H,A)$, $a \mapsto \kappa_a$ is obviously a homomorphism. Let $\varphi\colon H \to A$, $x^i \mapsto a^i$ for $i = 0,\dots,n-1$. Then

$$\partial\varphi(x^i,x^j) = \begin{cases} 1 & \text{if } i+j < n, \\ a^n & \text{if } i+j \geq n. \end{cases}$$

This shows $a^n \in \mathrm{Ker}(F)$. For $a \in \mathrm{Ker}(F)$ there exists a $\varphi \in C^1(H,A)$ with $\kappa_a = \partial\varphi$. It holds that

$$\begin{aligned} \varphi(1) &= \partial\varphi(1,1) = \kappa_a(1,1) = 1, \\ \varphi(x)^2 &= \kappa_a(x,x)\varphi(x^2) = \varphi(x^2), \\ \varphi(x)^3 &= \kappa_a(x^2,x)\varphi(x^3) = \varphi(x^3), \\ &\vdots \\ \varphi(x)^n &= \kappa_a(x^{n-1},x)\varphi(1) = a. \end{aligned}$$

Therefore $a \in \langle b^n : b \in A\rangle$ and $\mathrm{Ker}(F) = \langle a^n : a \in A\rangle$.

For the surjectivity of F, let $A \to G \xrightarrow{\pi} H$ be an extension of H by A. Let $\tilde{x} \in G$ with $\pi(\tilde{x}) = x$. For $i = 0,\dots,n-1$ we can choose $\widetilde{x^i} := \tilde{x}^i$ as a preimage of x^i. For the cocycle κ it then holds that

$$\kappa(x^i,x^j) = \begin{cases} \tilde{x}^i\tilde{x}^j\widetilde{x^{i+j}}^{-1} = 1 & \text{if } i+j < n, \\ \tilde{x}^i\tilde{x}^j\widetilde{x^{i+j-n}}^{-1} = \tilde{x}^n & \text{if } i+j \geq n. \end{cases}$$

With $a := \tilde{x}^n \in A$ it thus holds that $\kappa = \kappa_a$. Therefore F is surjective. □

Theorem 5.5. *For all abelian groups G and H it holds that $H_s^2(G\times H,A) \cong H_s^2(G,A) \times H_s^2(H,A)$.*

Proof. We follow the proof of the Künneth formula from group theory. We consider G and H as subgroups of $G \times H$. For $\kappa \in Z_s^2(G\times H,A)$ let $\kappa_G \in Z_s^2(G,A)$ be the restriction of κ to $G \times G$ and analogously $\kappa_H \in Z_s^2(H,A)$. Then

$$F\colon Z^2(G\times H,A) \to Z^2(G,A) \times Z^2(H,A), \qquad \kappa \mapsto (\kappa_G,\kappa_H)$$

is a homomorphism. For $\varphi \in C^1(G\times H,A)$ it is certain that $(\partial\varphi)_G = \partial\varphi_G \in B^2(G,A)$ and $(\partial\varphi)_H \in B^2(H,A)$. Thus F induces a homomorphism $\overline{F}\colon H_s^2(G\times H,A) \to H_s^2(G,A) \times H_s^2(H,A)$.

For the surjectivity of $\overline{F}$ let $\kappa_1 \in Z_s^2(G,A)$ and $\kappa_2 \in Z_s^2(H,A)$ be normalized. For $x_i \in G$ and $y_i \in H$ let $\kappa(x_1y_1, x_2y_2) := \kappa_1(x_1,x_2)\kappa_2(y_1,y_2)$. Then

$$\begin{aligned}\kappa(x_1y_1,x_2y_2)\kappa(x_1x_2y_1y_2,x_3y_3) &= \kappa_1(x_1,x_2)\kappa_2(y_1,y_2)\kappa_1(x_1x_2,x_3)\kappa_2(y_1y_2,y_3)\\ &= \kappa_1(x_2,x_3)\kappa_1(x_1,x_2x_3)\kappa_2(y_2,y_3)\kappa_2(y_1,y_2y_3)\\ &= \kappa(x_2y_2,x_3y_3)\kappa(x_1y_1,x_2y_2x_3y_3).\end{aligned}$$

This shows $\kappa \in Z_s^2(G\times H,A)$ with $\kappa_G = \kappa_1$ and $\kappa_H = \kappa_2$. Thus $\overline{F}$ is surjective.

For the injectivity let $F(\kappa) = (\partial\varphi_1, \partial\varphi_2)$ with $\varphi_1 \in C^1(G,A)$ and $\varphi_2 \in C^1(H,A)$. Let $\varphi \in C^1(G\times H, A)$ with $\varphi(xy) := \varphi_1(x)\varphi_2(y)\kappa(x,y)^{-1}$ for $x \in G$ and $y \in H$. Then

$$\begin{aligned}\partial\varphi(x_1y_1,x_2y_2) &= \varphi(x_1y_1)\varphi(x_2y_2)\varphi(x_1x_2y_1y_2)^{-1}\\ &= \varphi_1(x_1)\varphi_2(y_1)\kappa(x_1,y_1)^{-1}\varphi_1(x_2)\varphi_2(y_2)\kappa(x_2,y_2)^{-1}\varphi_1(x_1x_2)^{-1}\varphi_2(y_1y_2)^{-1}\kappa(x_1x_2,y_1y_2)\\ &= \kappa(x_1,x_2)\kappa(y_1,y_2)\kappa(x_1,y_1)^{-1}\kappa(x_2,y_2)^{-1}\kappa(x_1x_2,y_1y_2)\\ &= \kappa(x_2,y_1)\kappa(x_1,x_2y_1)\kappa(x_1x_2,y_1)^{-1}\kappa(y_1,y_2)\kappa(x_1,y_1)^{-1}\kappa(x_2,y_2)^{-1}\kappa(x_1x_2,y_1y_2)\\ &= \kappa(x_1y_1,x_2)\kappa(x_1x_2y_1,y_2)\kappa(x_2,y_2)^{-1} = \kappa(x_1y_1,x_2y_2)\end{aligned}$$

for $x_i \in G$ and $y_i \in H$. Thus $\kappa = \partial\varphi \in B^2(G\times H,A)$ and $\overline{F}$ is an isomorphism. □

Corollay 5.6. *For every finite abelian group A we have $H_s^2(A,\mathbb{C}^\times) = 1$.*

Proof. Since every complex number has an n-th root, $\langle z^n : z \in \mathbb{C}^\times\rangle = \mathbb{C}^\times$ holds. The claim follows from Theorem 5.5 and Theorem 5.4. □

Corollay 5.7. *For finite abelian groups $G \cong C_{d_1}\times\ldots\times C_{d_k}$ and $A \cong C_{e_1}\times\ldots\times C_{e_l}$ we have*

$$H_s^2(G,A) \cong \mathrm{Hom}(G,A) \cong \bigtimes_{\substack{1\le i\le k\\ 1\le j\le l}} C_{\gcd(d_i,e_j)}.$$

Proof. For $d \in \mathbb{N}$ we have $H^2(C_d,A) \cong A/\langle a^d : a\in A\rangle \cong \times_{i=1}^l C_{\gcd(d,e_i)}$ by Theorem 5.4. From Theorem 5.5 it follows that $H_s^2(G,A) \cong \times_{i,j} C_{\gcd(d_i,e_j)}$. Let $G = \langle x_1\rangle\times\ldots\times\langle x_k\rangle \cong C_{d_1}\times\ldots\times C_{d_k}$. Then $f \in \mathrm{Hom}(G,A)$ is uniquely determined by

$$f(x_i) \in \langle a\in A : a^{d_i} = 1\rangle \cong \bigtimes_{j=1}^{l} C_{\gcd(d_i,e_j)} \qquad (i = 1,\ldots,k)$$

and every choice defines a homomorphism $G \to A$. Clearly, the map $\mathrm{Hom}(G,A) \to \times_{i,j} C_{\gcd(d_i,e_j)}$, $f \mapsto (f(x_1),\ldots,f(x_k))$ is an isomorphism. The claim follows from this. □

Example 5.8. According to Corollay 5.7, there are four equivalence classes of abelian extensions of C_4 by C_4. However, there are only two isomorphism types of such groups: C_4^2 and C_{16}. In Theorem 5.19 we determine the structure of $H^2(G,A)$ for every finite group G.

Remark 5.9. For coprime $d_1,d_2 \in \mathbb{N}$ and $e \in \mathbb{N}$, $\gcd(d_1d_2,e) = \gcd(d_1,e)\gcd(d_2,e)$ holds. Therefore, the right side in Corollay 5.7 does not depend on the decomposition of G and A. Since the expression is symmetric, $H_s^2(A,B) \cong H_s^2(B,A)$ and $\mathrm{Hom}(A,B) \cong \mathrm{Hom}(B,A)$ hold for all finite abelian groups A and B.

Definition 5.10. An extension G of H by Z is called a *Schur extension*, if $Z \leq \mathrm{Z}(G) \cap G'$ holds. The *Schur multiplier* of H is $M(H) := H^2(H, \mathbb{C}^\times)$.

Example 5.11.

(i) Let G be a non-abelian finite nilpotent group. According to GT-Theorem 3.14, $G' \cap \mathrm{Z}(G) \neq 1$ holds. Therefore, G is a Schur extension of a smaller group. In particular, D_8 and Q_8 are Schur extensions of C_2^2. Furthermore, $G/G^{[3]}$ is a Schur extension of G/G'.

(ii) Every quasisimple group is a Schur extension of a simple group.

Remark 5.12.

(i) In the following, we study the possible Schur extensions of a fixed group H, which we rename to G for this purpose.

(ii) Let $\widehat{G}$ be a Schur extension of G with $1 \neq Z \leq \widehat{G}' \cap \mathrm{Z}(\widehat{G})$ and $\widehat{G}/Z \cong G$. If Z possesses a complement K in $\widehat{G}$, then $\widehat{G} = Z \times K$ and $Z \nleq 1 \times K' = \widehat{G}'$. Proper Schur extensions are therefore non-splitting.

(iii) If $n := |G| < \infty$, then $\exp(M(G)) \mid n$ according to Theorem 4.21. The values of $\kappa \in M(G)$ are thus n-th roots of unity. This shows $|M(G)| \leq |Z^2(G, \mathbb{C}^\times)| \leq n^{n^2} < \infty$. In the following, let G always be finite and A abelian.

(iv) According to Theorem 5.4, $M(C_n) = 1$ for all $n \in \mathbb{N}$.

(v) According to GT-Lemma 11.14, $A^* := \mathrm{Hom}(A, \mathbb{C}^\times) \cong A$, if $|A| < \infty$.

Lemma 5.13. *The map*

$$\Phi \colon H_s^2(G/G', A) \to H^2(G, A), \qquad \kappa B^2(G/G', A) \mapsto \Phi_\kappa B^2(G, A)$$

with $\Phi_\kappa(x, y) = \kappa(xG', yG')$ *for* $x, y \in G$ *is a monomorphism.*

Proof. Clearly $Z^2(G/G', A) \to Z^2(G, A)$, $\kappa \mapsto \Phi_\kappa$ is a homomorphism. For $\varphi \in C^1(G/G', A)$, $\Phi_{\partial\varphi} \in B^2(G, A)$ holds. Therefore Φ is well-defined on $H^2(G/G', A)$. Let $\kappa \in Z_s^2(G/G', A)$ and $\varphi \in C^1(G, A)$ with $\Phi_\kappa = \partial\varphi$. Let $R \subseteq G$ be a transversal for G/G'. For $x \in G$ let $r_x \in R$ with $xG' = r_xG'$. Let $\overline{\varphi} \in C^1(G/G', A)$ with $\overline{\varphi}(xG') := \varphi(r_x)$ for $x \in G$. Then

$$\kappa(xG', yG') = \kappa(r_xG', r_yG') = \Phi_\kappa(r_x, r_y) = \partial\varphi(r_x, r_y) = \partial\overline{\varphi}(xG', yG')$$

for $x, y \in G$. Therefore Φ is injective. □

Lemma 5.14. *The map*

$$\Psi \colon H^2(G, A) \to \mathrm{Hom}(A^*, M(G)), \qquad \kappa B^2(G, A) \mapsto \Psi_\kappa$$

with $\Psi_\kappa(\lambda) = (\lambda \circ \kappa)B^2(G, \mathbb{C}^\times)$ *is a homomorphism.*

Proof. For $\lambda, \mu \in A^*$, clearly $\lambda \circ \kappa \in Z^2(G, \mathbb{C}^\times)$ and $(\lambda\mu) \circ \kappa = (\lambda \circ \kappa)(\mu \circ \kappa)$. Thus $\Psi_\kappa \in \mathrm{Hom}(A^*, M(G))$. For $\varphi \in C^1(G, A)$ and $\lambda \in A^*$ we have

$$\Psi_{\partial\varphi}(\lambda) = (\lambda \circ \partial\varphi)B^2(G, \mathbb{C}^\times) = \partial(\lambda \circ \varphi)B^2(G, \mathbb{C}^\times) = 1.$$

Thus Ψ is well-defined. Due to $\lambda \circ (\kappa_1\kappa_2) = (\lambda \circ \kappa_1)(\lambda \circ \kappa_2)$, Ψ is a homomorphism. □

Theorem 5.15 (Schur)**.** *Let $\widehat{G}$ be a Schur extension of G with $\widehat{G}/Z \cong G$. Then Z is isomorphic to a subgroup of $M(G)$. In particular, $|\widehat{G}| \leq |G||M(G)|$ and G possesses only finitely many Schur extensions up to isomorphism.*

Proof. For $x \in G$ let $\widehat{x} \in \widehat{G}$ with $\widehat{x}Z = x$ and $\widehat{1} = 1$. Let $\kappa \in Z^2(G, Z)$ be the corresponding normalized factor system of $\widehat{G}$ (Lemma 4.5). It holds that $\kappa(x, y) = \widehat{x}\widehat{y}\widehat{xy}^{-1}$ for $x, y \in G$. Since $Z^* \cong Z$, it suffices to show that the map Ψ_κ from Lemma 5.14 is injective. So let $\lambda \in Z^*$ with $\lambda \circ \kappa = \partial\varphi$ for some $\varphi \in C^1(G, \mathbb{C}^\times)$. Then

$$\varphi(1) = \varphi(1)\varphi(1)\varphi(1)^{-1} = \partial\varphi(1) = \lambda(\kappa(1, 1)) = \lambda(1) = 1.$$

Let $\widehat{\lambda}\colon \widehat{G} \to \mathbb{C}^\times$ with $\widehat{\lambda}(\widehat{x}a) := \varphi(x)\lambda(a)$ for $x \in G$ and $a \in Z$. Because $\widehat{\lambda}(a) = \widehat{\lambda}(\widehat{1}a) = \lambda(a)$, $\widehat{\lambda}$ is an extension of λ. For $x, y \in G$ and $a, b \in Z$ it holds that

$$\begin{aligned}\widehat{\lambda}(\widehat{x}a \cdot \widehat{y}b) &= \widehat{\lambda}(\widehat{x}\widehat{y}ab) = \widehat{\lambda}(\kappa(x, y)\widehat{xy}ab) = \lambda(\kappa(x, y))\varphi(xy)\lambda(a)\lambda(b)\\ &= \varphi(x)\varphi(y)\varphi(xy)^{-1}\varphi(xy)\lambda(a)\lambda(b) = \varphi(x)\lambda(a)\varphi(y)\lambda(b) = \widehat{\lambda}(\widehat{x}a)\widehat{\lambda}(\widehat{y}b).\end{aligned}$$

Thus $\widehat{\lambda}$ is a homomorphism with $\widehat{G}/\mathrm{Ker}(\widehat{\lambda}) \leq \mathbb{C}^\times$. It follows that $Z \leq \widehat{G}' \leq \mathrm{Ker}(\widehat{\lambda})$. This shows $\lambda = 1$. □

Definition 5.16. A Schur extension $\widehat{G}$ of G is called *maximal*, if $|\widehat{G}| = |G||M(G)|$.

Theorem 5.17 (Schur)**.** *Every finite group G possesses a maximal Schur extension.*

Proof. According to Remark 5.12, $M(G) = \langle\overline{\kappa_1}\rangle \oplus \ldots \oplus \langle\overline{\kappa_n}\rangle$ is finite. Let $d_i := |\langle\overline{\kappa_i}\rangle|$ and $A_i \leq \mathbb{C}^\times$ with $|A_i| = d_i$ for $i = 1, \ldots, n$. Let $\kappa_i \in Z^2(G, \mathbb{C}^\times)$ with $\kappa_i B^2(G, \mathbb{C}^\times) = \overline{\kappa_i}$. Then $\kappa_i^{d_i} = \partial\gamma_i$ for some $\gamma_i \in C^1(G, \mathbb{C}^\times)$. Let $\delta_i(x) \in \mathbb{C}^\times$ with $\delta_i(x)^{d_i} = \gamma_i(x)^{-1}$ for $x \in G$. After replacing κ_i by $\kappa_i\partial\delta_i$, we have $\kappa_i^{d_i} = 1$ for $i = 1, \ldots, n$. In particular, $\kappa_i \in Z^2(G, A_i)$ for $i = 1, \ldots, n$. According to Lemma 4.11, we may also assume $\kappa_i(x, 1) = \kappa_i(1, x) = 1$ for $x \in G$. Let $A := A_1 \times \ldots \times A_n \cong M(G)$ and $\kappa \in C^2(G, A)$ with $\kappa(x, y) = (\kappa_1(x, y), \ldots, \kappa_n(x, y))$ for $x, y \in G$. Then clearly $\kappa \in Z^2(G, A)$ with $\kappa(1, x) = \kappa(x, 1) = 1$ for $x \in G$.

Let $\widehat{G} = A \times G$ be the central extension of A by G with respect to κ. By $a \mapsto (a, 1)$, we consider A as a subgroup of $\widehat{G}$. We choose preimages $\widehat{x} \in \widehat{G}$ of $x \in G$ such that $\kappa(x, y) = \widehat{x}\widehat{y}\widehat{xy}^{-1}$ holds for all $x, y \in G$. Let $\pi_i\colon A \to A_i \leq \mathbb{C}^\times$ be the i-th projection. With the map Ψ_κ from Lemma 5.14, we then have

$$\Psi_\kappa(\pi_i) = (\pi_i \circ \kappa)B^2(G, \mathbb{C}^\times) = \overline{\kappa_i}$$

for $i = 1, \ldots, n$. Because $M(G) = \langle\overline{\kappa_1}, \ldots, \overline{\kappa_n}\rangle$, Ψ_κ is surjective. According to Remark 5.12, $A^* \cong A \cong M(G)$. Therefore, Ψ_κ is also injective. According to the fundamental theorem of finite abelian groups (applied to $\widehat{G}/\widehat{G}'$), there exist normal subgroups $N_1, \ldots, N_s \trianglelefteq \widehat{G}$ with $\widehat{G}' = N_1 \cap \ldots \cap N_s$ such that $\widehat{G}/N_i$ is cyclic for $i = 1, \ldots, s$.

Assume $A \not\leq \widehat{G}'$. Then there exists an i with $A \not\leq N_i$. By embedding $\widehat{G}/N_i$ into $\mathbb{C}^\times$, one obtains a homomorphism $\lambda\colon \widehat{G} \to \widehat{G}/N_i \to \mathbb{C}^\times$ with $\lambda(A) \neq 1$. The restriction λ_A is thus a non-trivial element in A^*. For $x \in G$, we set $\varphi(x) := \lambda(\widehat{x})$. Then

$$\Psi_\kappa(\lambda_A)(x, y) = \lambda(\kappa(x, y)) = \lambda(\widehat{x}\widehat{y}\widehat{xy}^{-1}) = \varphi(x)\varphi(y)\varphi(xy)^{-1} = \partial\varphi(x, y)$$

for $x, y \in G$. This yields $\Psi_\kappa(\lambda_A) = 1$ in contradiction to the injectivity of Ψ_κ. Thus $A \leq \widehat{G}'$ and $\widehat{G}$ is a Schur extension of G. □

Remark 5.18. In GAP, there are several possibilities to determine the Schur multiplier and a maximal Schur extension:

```
G:=AlternatingGroup(7);;
AbelianInvariantsMultiplier(G); #orders of cyclic factors of M(G)
S:=SchurCover(G); #maximal Schur extension
S:=Image(IsomorphismPermGroup(S));; efficient representation as permutation group
NrMovedPoints(S); #degree of permutation group
S:=Image(SmallerDegreePermutationRepresentation(S));; #even more efficient representation
NrMovedPoints(S);
StructureDescription(S); #= C6.A7

epi:=EpimorphismSchurCover(G,[3]); #epimorphim Ĝ → G with kernel O3(M(G))
S:=Source(epi); #non-maximal Schur extension Ĝ
M:=Kernel(epi); #O3(M(G))

P:=SmallGroup(256,111);;
SchurCovers(P); #all maximal Schur extensions, only for p-groups

LoadPackage("cohomolo",false);
G:=PSL(3,4);;
for p in PrimeDivisors(Size(G)) do
  chr:=CHR(G,p);;
  Print(SchurMultiplier(chr)); #= Op(M(G)) faster than AbelianInvariantsMultiplier
od; #insgesmat folgt M(G) = O2(M(G)) ⊕ O3(M(G)) ≅ C12 × C4

LoadPackage("hap",false); #loads further packages
GroupHomology(G,2); #M(G) ≅ H2(G,ℤ)
```

The command `SchurExtension`, on the other hand, calculates the infinite central extension $F/[F,N]$ from Theorem 5.24.

Theorem 5.19 (Universal Coefficient Theorem[10])**.** *For every finite abelian group A,*

$$1 \longrightarrow H^2_s(G/G', A) \xrightarrow{\Phi} H^2(G, A) \xrightarrow{\Psi} \mathrm{Hom}(A^*, M(G)) \longrightarrow 1.$$

is an exact splitting sequence. In particular,

$$H^2(G, A) \cong \mathrm{Hom}(G/G' \times M(G), A).$$

Proof. The maps Φ and Ψ were defined in Lemma 5.13 and Lemma 5.14. Since Φ is injective, the sequence is exact at the first term. For $\kappa \in Z^2_s(G/G', A)$, we have $\lambda \circ \kappa \in Z^2_s(G/G', \mathbb{C}^\times) = 1$ by Corollay 5.6. This shows $\Psi_{\Phi_\kappa} = 1$ and $\Phi(H^2_s(G/G', A)) \leq \mathrm{Ker}(\Psi)$. Conversely, let $\kappa \in Z^2(G, A)$ with $\Psi_\kappa = 1$. Let $\widehat{G} := A \times G$ be the extension with respect to κ. By $a \mapsto (a, 1)$, we consider A as a subgroup of $\widehat{G}$. For $\lambda \in A^*$, there exists a $\varphi \in C^1(G, \mathbb{C}^\times)$ with $\lambda \circ \kappa = \partial\varphi$. We define $\widehat{\lambda}\colon \widehat{G} \to \mathbb{C}^\times$, $(a, x) \mapsto \lambda(a)\varphi(x)$. For $(a, x), (b, y) \in \widehat{G}$, it then holds that

$$\widehat{\lambda}\big((a, x)(b, y)\big) = \widehat{\lambda}\big(ab\kappa(x, y), xy\big) = \lambda(ab)\partial\varphi(x, y)\varphi(xy) = \lambda(a)\varphi(x)\lambda(b)\varphi(y) = \widehat{\lambda}(a, x)\widehat{\lambda}(b, y).$$

This shows $\widehat{\lambda} \in \widehat{G}^*$ and $\widehat{G}' \cap A \leq \mathrm{Ker}(\lambda)$. As is well known, for every $a \in A \setminus \{1\}$ there exists a $\lambda \in A^*$ with $\lambda(a) \neq 1$. It follows that $\widehat{G}' \cap A = 1$. Thus $\widehat{G}/\widehat{G}'$ is an extension of

$$\widehat{G}/\widehat{G}'A \cong (\widehat{G}/A)/(\widehat{G}'A/A) \cong G/G'$$

[10]The designation describes the fact that one can replace the codomain A of the cocycles with the *universal* codomain $\mathbb{C}^\times$.

with $A\widehat{G'}/\widehat{G'} \cong A/(A \cap \widehat{G'}) \cong A$. Let $\pi\colon \widehat{G} \to G$, $(a,x) \mapsto x$ be the projection and $\overline{\pi}\colon \widehat{G}/\widehat{G'} \to G/G'$, $x\widehat{G'} \mapsto \pi(x)G'$. Because of $\widehat{G'} \cong \widehat{G'}A/A \cong G'$, the restriction of π to $\widehat{G'}$ is injective. For $s \in G'$, let $\widehat{s} := \pi^{-1}(s) \in \widehat{G'}$. Let $R \subseteq G$ be a transversal for G/G'. For $x \in G$, let $r_x \in R$ with $xG' = r_xG'$. For $r \in R$, let $\widehat{r} \in \widehat{G}$ be an arbitrary preimage under π. For each $x \in G$, there exists exactly one $s_x \in G'$ with $x = r_xs_x$. Now $\widehat{x} := \widehat{r_x}\widehat{s_x}$ is a preimage of x under π. Therefore κ is equivalent to $\kappa' \in Z^2(G,A)$ with $\kappa'(x,y) = \widehat{x}\widehat{y}\widehat{xy}^{-1}$. Obviously $\widehat{x}\widehat{G'}$ is a preimage of xG' under $\overline{\pi}$ that does not depend on the choice of the representative of the coset xG'. Thus there exists a cocycle $\overline{\kappa} \in Z^2(G/G',A)$ with

$$\overline{\kappa}(xG',yG') = \widehat{x}\widehat{y}\widehat{xy}^{-1}\widehat{G'} = \kappa'(x,y)\widehat{G'}.$$

This shows $\kappa B^2(G,A) = \kappa' B^2(G,A) = \Phi_{\overline{\kappa}}$ by identifying A with $A\widehat{G'}/\widehat{G'}$. Since $\widehat{G}/\widehat{G'}$ is abelian, $\overline{\kappa} \in Z_s^2(G/G',A)$ and $\Phi(H_s^2(G/G',A)) = \mathrm{Ker}(\Psi)$ holds. Thus the sequence is also exact at the second term.

We now construct a homomorphism $\Gamma\colon \mathrm{Hom}(A^*, M(G)) \to H^2(G,A)$ with $\Psi \circ \Gamma = \mathrm{id}$. From this, one obtains both the surjectivity of Ψ (exactness at the third term) and the splitting of the sequence. Let $Z := M(G)$ and $\kappa \in Z^2(G,Z)$ be the cocycle of a maximal Schur extension. In the proof of Theorem 5.17, we have seen that the map $Z^* \to Z$, $\lambda \mapsto (\lambda \circ \kappa)B^2(G,\mathbb{C}^\times)$ is an isomorphism. Let $f \in \mathrm{Hom}(A^*,Z)$. For $\mu \in A^*$, there exists exactly one $\lambda_\mu \in Z^*$ with $f(\mu) = (\lambda_\mu \circ \kappa)B^2(G,\mathbb{C}^\times)$. For $x,y \in G$, we define $\alpha_f(x,y)\colon A^* \to \mathbb{C}^\times$, $\mu \mapsto \lambda_\mu(\kappa(x,y))$. From

$$(\lambda_{\mu_1\mu_2} \circ \kappa)B^2(G,\mathbb{C}^\times) = f(\mu_1\mu_2) = f(\mu_1)f(\mu_2) = (\lambda_{\mu_1} \circ \kappa)(\lambda_{\mu_2} \circ \kappa)B^2(G,\mathbb{C}^\times) = (\lambda_{\mu_1}\lambda_{\mu_2} \circ \kappa)B^2(G,\mathbb{C}^\times)$$

it follows that $\lambda_{\mu_1\mu_2} = \lambda_{\mu_1}\lambda_{\mu_2}$. This shows $\alpha_f(x,y) \in (A^*)^*$. Because $|A| < \infty$, the map $A \to (A^*)^*$, $a \mapsto (\mu \mapsto \mu(a))$ is an isomorphism (GT-Exercise 79). Therefore there exists exactly one $\Gamma_f(x,y) \in A$ with $\alpha_f(x,y)(\mu) = \mu(\Gamma_f(x,y))$ for all $\mu \in A^*$. For $x,y,z \in G$ and $\mu \in A^*$, it holds that

$$\begin{aligned}\mu\big(\Gamma_f(x,y)\Gamma_f(xy,z)\big) &= \alpha_f(x,y)(\mu)\alpha_f(xy,z)(\mu) = \lambda_\mu\big(\kappa(x,y)\kappa(xy,z)\big)\\ &= \lambda_\mu\big(\kappa(y,z)\kappa(x,yz)\big) = \mu\big(\Gamma_f(y,z)\Gamma_f(x,yz)\big).\end{aligned}$$

This shows $\Gamma_f \in Z^2(G,A)$. For $f' \in \mathrm{Hom}(A^*,Z)$ and $\mu \in A^*$, let $\lambda'_\mu \in Z^*$ with $f'(\mu) = (\lambda'_\mu \circ \kappa)B^2(G,\mathbb{C}^\times)$. Then $(ff')(\mu) = (\lambda_\mu\lambda'_\mu \circ \kappa)B^2(G,\mathbb{C}^\times)$ and

$$\mu\big(\Gamma_{ff'}(x,y)\big) = \alpha_{ff'}(x,y)(\mu) = (\lambda_\mu\lambda'_\mu)(\kappa(x,y)) = \alpha_f(x,y)(\mu)\alpha_{f'}(x,y)(\mu) = \mu\big(\Gamma_f(x,y)\Gamma_{f'}(x,y)\big).$$

Thus Γ is a homomorphism. From

$$\Psi_{\Gamma_f}(\mu)(x,y) = (\mu \circ \Gamma_f)(x,y) = \alpha_f(x,y)(\mu) = \lambda_\mu(\kappa(x,y)) = f(\mu)(x,y).$$

it follows that $\Psi \circ \Gamma = \mathrm{id}$.

For the second assertion, we use Remark 5.9 and $A^* \cong A$:

$$H^2(G,A) \cong H_s^2(G/G',A) \times \mathrm{Hom}(A^*,Z) \cong \mathrm{Hom}(G/G',A) \times \mathrm{Hom}(Z,A) \cong \mathrm{Hom}(G/G' \times Z, A). \quad \square$$

Theorem 5.20. *Let $G/G' \cong C_{d_1} \times \cdots \times C_{d_k}$ and $M(G) \cong C_{e_1} \times \cdots \times C_{e_l}$. Then G has at most*

$$\prod_{\substack{1 \le i \le k \\ 1 \le j \le l}} \gcd(d_i,e_j)$$

maximal Schur extensions up to isomorphism.

Proof. Let $\widehat{G} := Z \times G$ be a maximal Schur extension of G with respect to $\kappa \in Z^2(G,Z)$ with $Z \cong M(G)$. As in the proof of Theorem 5.17, $\Psi_\kappa \colon Z^* \to M(G)$ is an isomorphism. We show that the isomorphism type of $\widehat{G}$ does not depend on Ψ_κ. For this, let $f \in \mathrm{Aut}(Z)$ and $\kappa' \in Z^2(G,Z)$ with $\kappa'(x,y) = f(\kappa(x,y))$ for all $x,y \in G$. Let $\widehat{H} := Z \times G$ be the extension of G by Z with respect to κ'. Then

$$F \colon \widehat{G} \to \widehat{H}, \qquad (a,x) \mapsto (f(a),x)$$

is an isomorphism, because

$$\begin{aligned} F\big((a,x)(b,y)\big) &= F\big(ab\kappa(x,y), xy\big) = \big(f(ab\kappa(x,y)), xy\big) \\ &= \big(f(a)f(b)\kappa'(x,y), xy\big) = (f(a),x)(f(b),y) = F(a,x)F(b,y) \end{aligned}$$

for $(a,x),(b,y) \in \widehat{G}$. Let $f^* \in \mathrm{Aut}(Z^*)$, $\lambda \mapsto \lambda \circ f$ be the dual map to f. Then $\Psi_{\kappa'} = \Psi_\kappa \circ f^*$ holds. In this way, every isomorphism $Z^* \to M(G)$ can be realized. For every further maximal Schur extension of G with respect to an $\alpha \in Z^2(G,Z)$, one can therefore assume $\Psi_\alpha = \Psi_\kappa$. The number of maximal Schur extensions is thus bounded by $|\Psi^{-1}(\Psi_\kappa)| = |\mathrm{Ker}(\Psi)| = |H_s^2(G/G',Z)|$ according to Theorem 5.19. Corollay 5.7 yields the assertion. □

Example 5.21.

(i) Let p be a prime and $G \cong C_p^n$ elementary abelian. According to GT-Example 11.38, $M(G) \cong C_p^{\binom{n}{2}}$. Thus G has at most $p^{\binom{n}{2}}$ maximal Schur extensions up to isomorphism. This can also be proven directly: Let $\widehat{G}$ be a maximal Schur extension and $Z \leq \widehat{G}' \cap \mathrm{Z}(\widehat{G})$ with $\widehat{G}/Z \cong G$. Because of $Z \leq \Phi(\widehat{G})$, G can be generated by n elements $x_1,\ldots,x_n$. Then $\{[x_i,x_j] : 1 \leq i < j \leq n\}$ is a basis of $Z \cong M(G)$. For $i = 1,\ldots,n$ there exist $0 \leq e_{ijk} < p$ with

$$x_i^p = \prod_{j<k} [x_j,x_k]^{e_{ijk}}.$$

Because of $Z \leq \mathrm{Z}(\widehat{G})$, $\widehat{G}$ is uniquely determined by the $n\binom{n}{2}$ parameters e_{ijk}. Obviously, many parameter values lead to isomorphic groups (for example by permutation of the x_i). In Remark 12.36 we construct the maximal Schur extension with $p > 2$ and $e_{ijk} = 0$ for all i,j,k. For $n = 2$, the two non-abelian groups of order p^3 are the only maximal Schur extensions. For $p^n \in \{2^3, 2^4, 3^3, 5^3\}$ there are 10, 989, 16 and 20 maximal Schur extensions of G, respectively. In general, the number is unknown.

(ii) The minimal non-abelian 2-group $Q(2,1)$ from Exercise 6 has exactly seven maximal Schur extensions:

```
G:=SmallGroup(16,3);; #=Q(2,1)
ab:=AbelianInvariants(G); #=[2,4]
mG:=AbelianInvariantsMultiplier(G); #=[2,2]
ProductX(ab,mG,{d,e}->Gcd(d,e)); #=16
Size(SchurCovers(G)); #=7
```

Remark 5.22.

(i) If G is perfect (or more generally $\gcd(|G/G'|, |M(G)|) = 1$), then G has only one maximal Schur extension $\widehat{G}$ up to isomorphism. One calls $\widehat{G}$ the *universal* Schur extension of G, because according to Theorem 5.24 every Schur extension of G is a quotient of $\widehat{G}$.

(ii) For complete groups G, equality holds in Theorem 5.20 (without proof).

Theorem 5.23. *Let G be any group with $|G : \mathrm{Z}(G)| < \infty$. Then $|G'| < \infty$.*

Proof (*Rosenlicht*). Let $Z := \mathrm{Z}(G)$ and $n := |G : Z| < \infty$. Let R be a transversal for G/Z. For $\Gamma := \{[r, s] : r, s \in R\}$ it holds that $|\Gamma| \leq |R|^2 = |G/Z|^2 = n^2$. For $r, s \in R$ and $z \in Z$ it holds that $[rz, s] = [r, s] = [r, sz]$. Every element $g \in G'$ thus has the form $g = c_1 \dots c_m$ with $c_1, \dots, c_m \in \Gamma$. It suffices to show that one can choose $m \leq n^3$ (then it follows that $|G'| \leq n^{2n^3} < \infty$). Assume $m > n^3$. Then there exists a $\gamma \in \Gamma$ with $|\{i \in \{1, \dots, m\} : c_i = \gamma\}| > n$. Because of $c_i c_{i+1} = c_{i+1}(c_{i+1}^{-1} c_i c_{i+1}) = c_{i+1}\delta$ with $\delta \in \Gamma$ we can assume $c_1 = \ldots = c_{n+1} = \gamma$. In contradiction to the minimality of m we will show that γ^{n+1} is a product of n commutators. For this, let $\gamma = [r, s]$ with $r, s \in R$. Because of $\gamma^n = \gamma^{|G:Z|} \in Z$ we have

$$\gamma^{n+1} = \gamma\gamma^n = \gamma s \gamma^n s^{-1} = \gamma s \gamma s^{-1} (s\gamma s^{-1})^{n-1} = [r, s]s[r, s]s^{-1}[srs^{-1}, s]^{n-1} = [r, s^2][srs^{-1}, s]^{n-1}. \quad \square$$

Theorem 5.24. *Let $G = F/N$ be a finite group with $F = F_n$. Then:*

(i) *$N/[F, N]$ is a finitely generated abelian group with free part of rank n and torsion part $(F' \cap N)/[F, N]$.*

(ii) *For $N/[F, N] = (F' \cap N)/[F, N] \oplus K/[F, N]$, F/K is a maximal Schur extension of G.*

(iii) *For every Schur extension $\widehat{G}$ of G there exists an $L \trianglelefteq F$ with $N = (F' \cap N)L$ and $\widehat{G} \cong F/L$. In particular, $\widehat{G}$ is a factor group of a maximal Schur extension.*

(iv) $\boxed{M(G) \cong (F' \cap N)/[F, N]}$ (Hopf formula).

Proof.

(i) According to Theorem 2.11, N is finitely generated. With $N \trianglelefteq F$, we have $[F, N] \trianglelefteq F$ and $[F, N] \leq F' \cap N$. Because $N/[F, N] \leq \mathrm{Z}(F/[F, N])$, $\mathrm{Z}(F/[F, N])$ has finite index in $F/[F, N]$. According to Theorem 5.23, $F'/[F, N]$ is finite. Therefore, $(F' \cap N)/[F, N]$ is also finite. Because $N' \leq [F, N]$, $N/[F, N]$ is abelian. Furthermore,

$$(N/[F, N])/((F' \cap N)/[F, N]) \cong N/(F' \cap N) \cong NF'/F' \leq F/F'.$$

According to Example 1.17, F/F' is a free abelian group of rank n. Because $|F/N| = |G| < \infty$, NF'/F' must also be a free abelian group of rank n. Therefore, $(F' \cap N)/[F, N]$ is the torsion part of $N/[F, N]$.

(ii) Because $K/[F, N] \leq N/[F, N] \leq \mathrm{Z}(F/[F, N])$, we have $K \trianglelefteq F$. Let $\widehat{G} := F/K$ and $Z := N/K$. Then $\widehat{G}/Z \cong F/N \cong G$ and $Z \leq \mathrm{Z}(\widehat{G})$ because of $[F, N] \leq K$. From $N/[F, N] \leq F'K/[F, N]$ it follows that

$$Z = N/K \leq F'K/K = (F/K)' = \widehat{G}'.$$

Thus $\widehat{G}$ is a Schur extension with $Z \cong (F' \cap N)/[F, N]$. From Theorem 5.15 it follows that $|M(G)| \geq |(F' \cap N)/[F, N]|$. For the reverse inequality, we first show (iii).

(iii) Let $\alpha \colon F \to G$ and $\beta \colon \widehat{G} \to G$ be the canonical epimorphisms with $N = \mathrm{Ker}(\alpha)$ and $Z := \mathrm{Ker}(\beta)$. Since F is free, there exists a homomorphism $\rho \colon F \to \widehat{G}$ with $\beta\rho = \alpha$. It then holds that $\widehat{G} = \rho(F)Z$ and $Z \leq \widehat{G}' \leq \rho(F)' \leq \rho(F)$, thus $\rho(F) = \widehat{G}$. Obviously $L := \mathrm{Ker}(\rho) \leq \mathrm{Ker}(\alpha) = N$. Because $\beta\rho(N) = \alpha(N) = 1$, we have $\rho(N) \leq \mathrm{Ker}(\beta) = Z$. This shows $\rho([F, N]) \leq [\widehat{G}, Z] = 1$ and $[F, N] \leq L$. From $\rho(N) = Z \leq \widehat{G}' = \rho(F')$ it also follows that $N \leq F'L$ and $(F' \cap N)L = F'L \cap N = N$ by Dedekind. Now it holds that

$$|Z| = \frac{|\widehat{G}|}{|G|} = |N : L| = |(F' \cap N)L : L| = |F' \cap N : F' \cap L| \leq |(F' \cap N)/[F, N]|.$$

Therefore, the Schur extension constructed in (ii) is indeed maximal.

According to (i) and the fundamental theorem of finitely generated abelian groups, it holds that

$$L/[F,N] = (L \cap F')/[F,N] \oplus M/[F,N],$$

where $(L \cap F')/[F,N]$ is the torsion part. Because $|N : L| = |F' \cap N : F' \cap L|$, $M/[F,N]$ is the torsion-free part of $N/[F,N]$. According to (ii), F/M is a maximal Schur extension and $\widehat{G} \cong F/L \cong (F/M)/(L/M)$.

(iv) Follows from the proof of (ii). □

Theorem 5.25. *Let $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$ be finite. Then $M(G)$ can be generated with $k - n$ elements. In particular, $M(G) = 1$ if $n = k$.*

Proof. According to Theorem 1.21, $n \leq k$. Let F be the free group w.r.t. $x_1, \ldots, x_n$ and $N := \langle r_1, \ldots, r_k \rangle^F$. Because of $N/[F,N] \leq \mathrm{Z}(F/[F,N])$, we have $N/[F,N] = \langle \overline{r_1}, \ldots, \overline{r_k} \rangle$, where $\overline{r_i} := r_i[F,N]$. According to Theorem 5.24, the free part of $N/[F,N]$ has rank n and the torsion part is isomorphic to $M(G)$. Thus $M(G)$ can be generated with $k - n$ elements. □

Example 5.26. According to Theorem 5.25 and Exercise 2, $M(Q_{2^n}) = 1$ for $n \geq 3$ (cf. GT-Example 11.38). For infinite groups, Theorem 5.25 is false: For example, $F_2/F_2^{[3]}$ is a Schur extension of $F_2/F_2' \cong \langle x, y \mid [x,y] = 1 \rangle \cong C_\infty^2$.

Theorem 5.27. *Let $\widehat{G}_1$, $\widehat{G}_2$ be maximal Schur extensions of G with $\widehat{G}_1/Z_1 \cong G \cong \widehat{G}_2/Z_2$. Then*

(i) (Schur) *$\widehat{G}_1' \cong \widehat{G}_2'$ and $\widehat{G}_1/\widehat{G}_1' \cong G/G' \cong \widehat{G}_2/\widehat{G}_2'$.*

(ii) (Gaschütz) *$\widehat{G}_1/\mathrm{Z}(\widehat{G}_1) \cong \widehat{G}_2/\mathrm{Z}(\widehat{G}_2)$.*

(iii) (Read) *$\mathrm{Z}(\widehat{G}_1)/Z_1 \cong \mathrm{Z}(\widehat{G}_2)/Z_2$.*

Proof. Let $G = F/N$, $K_i \trianglelefteq F$ and $\widehat{G}_i \cong F/K_i$ as in Theorem 5.24. We show that the specified groups do not depend on i.

(i) It holds that

$$\begin{aligned} \widehat{G}_i' &\cong F'K_i/K_i \cong F'/(F' \cap K_i) = F'/(F' \cap N \cap K_i) = F'/[F,N], \\ \widehat{G}_i/\widehat{G}_i' &\cong (\widehat{G}_i/Z_i)/(\widehat{G}_i'/Z_i) \cong G/G'. \end{aligned}$$

(ii) For $L/[F,N] := \mathrm{Z}(F/[F,N])$, we have $[F,L] \leq [F,N] \leq K_i$ and $L/K_i \leq \mathrm{Z}(F/K_i)$. Conversely, let $xK_i \in \mathrm{Z}(F/K_i)$. Then $[x,F] \leq K_i \cap F' = K_i \cap F' \cap N = [F,N]$ and it follows that $x[F,N] \in \mathrm{Z}(F/[F,N]) = L/[F,N]$. This shows

$$\widehat{G}_i/\mathrm{Z}(\widehat{G}_i) \cong (F/K_i)/\mathrm{Z}(F/K_i) = (F/K_i)/(L/K_i) \cong F/L.$$

(iii) With the notation from (ii), it holds that

$$\mathrm{Z}(\widehat{G}_i)/Z_i \cong (L/K_i)/(N/K_i) \cong L/N.$$ □

Remark 5.28.

(i) In general, $\mathrm{Z}(\widehat{G}_1) \ncong \mathrm{Z}(\widehat{G}_2)$ in the situation of Theorem 5.27. For example, the minimal non-abelian groups

$$\widehat{G}_1 = \langle x, y \mid x^8 = y^2 = 1,\ yxy^{-1} = x^5 \rangle,$$
$$\widehat{G}_2 = \langle x, y \mid x^4 = y^2 = [x,y]^2 = [x,x,y] = [y,x,y] = 1 \rangle$$

of order 16 from Exercise 4 and Exercise 6 are maximal Schur extensions of $G = C_4 \times C_2$ with $\mathrm{Z}(\widehat{G}_1) = \langle x^2 \rangle \cong C_4$ and $\mathrm{Z}(\widehat{G}_2) = \langle x^2, [x,y] \rangle \cong C_2^2$ (from GT-Theorem 11.37 it follows that $M(G) \cong C_2$).

(ii) Schur multipliers also occur in the representation theory of finite groups: Let $N \trianglelefteq G$ and $\Delta\colon N \to \mathrm{GL}(d,\mathbb{C})$ be an irreducible representation of N. For $g \in G$, ${}^g\Delta\colon N \to \mathrm{GL}(d,\mathbb{C})$, $x \mapsto \Delta(g^{-1}xg)$ is also a representation. Let us assume that Δ and ${}^g\Delta$ are equivalent for all $g \in G$, i. e. for $g \in G$ there exists $T_g \in \mathrm{GL}(d,\mathbb{C})$ with ${}^g\Delta(x) = T_g \Delta(x) T_g^{-1}$ for all $x \in N$. Let $g_1, \ldots, g_k \in G$ be a transversal for G/N and $T_i := T_{g_i}$. For $x \in N$ we define

$$\Gamma\colon G \to \mathrm{GL}(d,\mathbb{C}), \quad g_i x \mapsto \Delta(x^{-1}) T_i.$$

For $y \in N$ we have

$$\Gamma(g_i x)\Delta(y)\Gamma(g_i x)^{-1} = \Delta(x^{-1})T_i\Delta(y)T_i^{-1}\Delta(x) = \Delta(x^{-1}g_i^{-1}yg_ix) = {}^{g_ix}\Delta(y).$$

This shows

$$\Gamma(gh)\Delta(y)\Gamma(gh)^{-1} = {}^{gh}\Delta(y) = {}^g({}^h\Delta)(y) = \Gamma(g)\Gamma(h)\Delta(y)\Gamma(h)^{-1}\Gamma(g)^{-1}$$

for $g, h \in G$ and all $y \in N$. From Schur's Lemma[11] it follows that $\Gamma(gh) = \kappa(g,h)\Gamma(g)\Gamma(h)$ for some $\kappa(g,h) \in \mathbb{C}^\times$. One calls Γ a *projective representation* of G, because $G \to \mathrm{PGL}(d,\mathbb{C})$, $g \mapsto \Gamma(g)\mathbb{C}^\times$ is a homomorphism. Because of

$$\begin{aligned}\kappa(x,yz)\kappa(y,z)\Gamma(x)\Gamma(y)\Gamma(z) &= \kappa(x,yz)\Gamma(x)\Gamma(yz) = \Gamma(x(yz)) = \Gamma((xy)z)\\ &= \kappa(xy,z)\Gamma(xy)\Gamma(z) = \kappa(xy,z)\kappa(x,y)\Gamma(x)\Gamma(y)\Gamma(z)\end{aligned}$$

we have $\kappa \in Z^2(G,\mathbb{C}^\times)$. In fact, κ depends only on G/N, i. e. $\kappa \in Z^2(G/N,\mathbb{C}^\times)$. Other choices of T_i yield equivalent κ. In the case $M(G/N) = 1$, one can achieve $\kappa = 1$. Then Γ is an ordinary representation that extends Δ.

(iii) We now concern ourselves with the question of which groups have the form $G/\mathrm{Z}(G)$. In doing so, we again allow infinite groups G.

Definition 5.29. One calls G *capable*, if a group H with $H/\mathrm{Z}(H) \cong G$ exists. Let $\mathrm{Z}^*(G)$ be the intersection of all normal subgroups N such that G/N is capable. One calls $\mathrm{Z}^*(G)$ the *epicentre* of G.

Remark 5.30.

(i) Capable groups are inner automorphism groups because of $H/\mathrm{Z}(H) \cong \mathrm{Inn}(H)$.

(ii) Since $G/\mathrm{Z}(G)$ is capable, $\mathrm{Z}^*(G) \leq \mathrm{Z}(G)$ holds. Obviously, $\mathrm{Z}^*(G)$ is characteristic in G.

[11] see Lemma 1.9 in character theory notes

Example 5.31. Groups with trivial center are capable. Because of $D_{4n}/\mathrm{Z}(D_{4n}) \cong D_{2n}$, all dihedral groups are capable. Non-trivial cyclic groups are not capable.

Lemma 5.32. *For every group G, $G/\mathrm{Z}^*(G)$ is capable.*

Proof. Let $\{N_i : i \in I\}$ be the set of normal subgroups of G such that G/N_i is capable. Then there exist groups $\{H_i : i \in I\}$ and epimorphisms $\varphi_i \colon H_i \to G/N_i$ with $\mathrm{Ker}(\varphi_i) = \mathrm{Z}(H_i)$. Let $H := \times_{i\in I} H_i$ and

$$L := \big\{(h_i)_i \in H : \exists g \in G : \forall i \in I : \varphi_i(h_i) = gN_i\big\} \leq H.$$

Certainly $\mathrm{Z}(H) = \times \mathrm{Z}(H_i) \leq \mathrm{Z}(L)$ holds. Since the projection from L to H_i is surjective for every i, $\mathrm{Z}(H) = \mathrm{Z}(L)$ holds. For every $g \in G$ there exists exactly one tuple $(h_i)_i\mathrm{Z}(L) \in L/\mathrm{Z}(L)$ with $\varphi_i(h_i) = gN_i$ for all $i \in I$. One easily sees that the map $G \to L/\mathrm{Z}(L)$, $g \mapsto (h_i)_i\mathrm{Z}(L)$ is an epimorphism with kernel $\mathrm{Z}^*(G) = \bigcap_{i\in I} N_i$. Thus $G/\mathrm{Z}^*(G) \cong L/\mathrm{Z}(L)$ is capable. □

Corollay 5.33. *G is capable if and only if $\mathrm{Z}^*(G) = 1$ holds.*

Lemma 5.34. *For every group G, $\mathrm{Z}^*(G)$ is the intersection of all subgroups of the form $\pi(\mathrm{Z}(H))$, where $Z \to H \xrightarrow{\pi} G$ is a central extension of G.*

Proof. Let $D \leq G$ be the intersection over $\pi(\mathrm{Z}(H))$ as specified. For every central extension $\pi \colon H \to G$, $\pi(\mathrm{Z}(H)) \trianglelefteq \pi(H) = G$ and

$$G/\pi(\mathrm{Z}(H)) \cong (H/\mathrm{Ker}(\pi))/(\mathrm{Z}(H)/\mathrm{Ker}(\pi)) \cong H/\mathrm{Z}(H)$$

hold. Thus $G/\pi(\mathrm{Z}(H))$ is capable and $\mathrm{Z}^*(G) \leq D$.

Conversely, let G/N be capable for some $N \trianglelefteq G$. Then there exists an epimorphism $\alpha \colon H \to G/N$ with $\mathrm{Ker}(\alpha) = \mathrm{Z}(H)$. Let

$$L := \big\{(x, y) \in G \times H : \alpha(y) = xN\big\} \leq G \times H.$$

The projection $\rho \colon L \to G$, $(x, y) \to x$ is surjective with $\mathrm{Ker}(\rho) = 1 \times \mathrm{Ker}(\pi) \leq \mathrm{Z}(L)$. Thus ρ is a central extension of G. It follows that $D \leq \rho(\mathrm{Z}(L))$. Since the projection from L to H is surjective, $\mathrm{Z}(L) \leq G \times \mathrm{Z}(H)$ holds. Because of $\mathrm{Z}(H) = \mathrm{Ker}(\alpha)$, $\mathrm{Z}(L) \leq N \times \mathrm{Z}(H)$ and $D \leq \rho(\mathrm{Z}(L)) \leq N$. This shows $D \leq \mathrm{Z}^*(G)$. □

Remark 5.35. We show that in Lemma 5.34 (for finite groups) only a single central extension needs to be considered.

Theorem 5.36 (Beyl-Felgner-Schmid)**.** *Let $\widehat{G}$ be a maximal Schur extension of a finite group G and $\gamma \colon \widehat{G} \to G$ the corresponding epimorphism. Then $\mathrm{Z}^*(G) = \gamma(\mathrm{Z}(\widehat{G}))$ holds.*

Proof. Since every Schur extension is central, $\mathrm{Z}^*(G) \leq \gamma(\mathrm{Z}(\widehat{G}))$ holds by Lemma 5.34. Conversely, let $\alpha \colon H \to G$ be a central extension. We must show $\gamma(\mathrm{Z}(\widehat{G})) \leq \alpha(\mathrm{Z}(H))$. Let F be a free group and $\pi \colon F \to G$ an epimorphism with kernel N. Then there exists a homomorphism $\varphi \colon F \to H$ with $\alpha\varphi = \pi$. Because of $\alpha(\varphi(N)) = \pi(N) = 1$, we have $\varphi(N) \leq \mathrm{Ker}(\alpha) \leq \mathrm{Z}(H)$. It follows that $\varphi([F, N]) \leq [H, \mathrm{Z}(H)] = 1$. Since π is surjective, $H = \mathrm{Ker}(\alpha)\varphi(F) = \mathrm{Z}(H)\varphi(F)$ holds.

According to Theorem 5.24, we can assume $\widehat{G} = F/K$, where $N/[F, N] = (F' \cap N)/[F, N] \oplus K/[F, N]$. Furthermore, let $\gamma(xK) = \pi(x)$ for all $x \in F$. Let $W/K := \mathrm{Z}(\widehat{G})$. Then

$$[\varphi(W), \varphi(F)] = \varphi([W, F]) \leq \varphi(F' \cap K) = \varphi(F' \cap N \cap K) = \varphi([F, N]) = 1.$$

It follows that $\varphi(W) \leq \mathrm{Z}(H)$ and $\gamma(\mathrm{Z}(\widehat{G})) = \pi(W) \leq \alpha(\mathrm{Z}(H))$. □

Corollay 5.37. *Let G be finite with $M(G) = 1$. G is centrally extendable if and only if $\mathrm{Z}(G) = 1$ holds.*

Example 5.38. For $n \geq 3$, Q_{2^n} is not centrally extendable, because $M(Q_{2^n}) = 1$ (Example 5.26) and $\mathrm{Z}(Q_{2^n}) \neq 1$.

Theorem 5.39 (Baer)**.** *Let $A = C_{d_1} \times \ldots \times C_{d_k}$ be a finite abelian group with $1 \neq d_1 \mid d_2 \mid \ldots \mid d_k$. A is centrally extendable if and only if $k \geq 2$ and $d_{k-1} = d_k$ holds.*

Proof. For $k = 1$, $A \neq 1$ is cyclic and therefore not centrally extendable. Now let $k \geq 2$. Let $A = \langle a_1 \rangle \times \ldots \times \langle a_k \rangle$ with $|\langle a_i \rangle| = d_i$ for $i = 1, \ldots, k$. Assume first that A is centrally extendable. Let $\gamma\colon H \to A$ be an epimorphism with $\mathrm{Ker}(\gamma) = \mathrm{Z}(H)$. Let $H_i := \gamma^{-1}(\langle a_i a_k \rangle)$ for $i = 1, \ldots, k-1$ and $H_k := \gamma^{-1}(\langle a_k \rangle)$. Since $H_i/\mathrm{Z}(H)$ is cyclic, H_i is abelian for $i = 1, \ldots, k$. Because $A = \langle a_1 a_k, \ldots, a_{k-1} a_k, a_k \rangle$, it holds that $H = \langle H_1, \ldots, H_k \rangle$ and $\bigcap_{i=1}^{k} H_i = \mathrm{Z}(H)$. Because $(a_i a_k)^{d_{k-1}} = a_k^{d_{k-1}}$ for $i = 1, \ldots, k-1$, we have $\gamma^{-1}(a_k^{d_{k-1}}) \in \mathrm{Z}(H) = \mathrm{Ker}(\gamma)$, d. h. $d_k = |\langle a_k \rangle| = d_{k-1}$.

Now let $Z := \mathrm{Z}^*(A) \neq 1$. Let F be a free group and $N, L \trianglelefteq F$ with $F/N \cong A$ and $F/L \cong (F/N)/(L/N) \cong A/Z$. Let $\widehat{A} := F/K$ be a maximal Schur extension of A with $N/[F,N] = (F' \cap N)/[F,N] \oplus K/[F,N]$. According to Theorem 5.36, it holds that $\mathrm{Z}(\widehat{A}) = L/K$ and $[F,L] \leq F' \cap K \leq [F,N]$. Therefore, the map

$$M(A) = (F' \cap N)/[F,N] \to (F' \cap L)/[F,L] = M(A/Z), \qquad x[F,N] \mapsto x[F,L]$$

is injective. In particular, $|M(A)| \leq |M(A/Z)|$. According to GT-Example-11.38, $|M(A)| = d_1^{k-1} d_2^{k-2} \ldots d_{k-1}$ holds. On the other hand, there exist $e_i \mid d_i$ with $M(A/Z) = e_1^{k-1} \ldots e_{k-1}$. From $|M(G)| \leq |M(A/Z)|$ it now follows easily that $d_{k-1} < d_k$. □

Definition 5.40. Groups G and H are called *isoclinic*, if there are isomorphisms

$$\varphi\colon G' \to H', \qquad \psi\colon G/\mathrm{Z}(G) \to H/\mathrm{Z}(H), \quad x\mathrm{Z}(G) \mapsto \tilde{x}\mathrm{Z}(H)$$

with $\varphi([x,y]) = [\tilde{x}, \tilde{y}]$ for all $x, y \in G$. If applicable, ψ is called an *isoclinism* and we write $G \approx H$.

Remark 5.41.

(i) The property $\varphi([x,y]) = [\tilde{x}, \tilde{y}]$ does not depend on the choice of the representatives $\tilde{x}$. Furthermore, $\varphi(g)\mathrm{Z}(H) = \psi(g\mathrm{Z}(G))$ holds for all $g \in G'$.

(ii) Obviously, isoclinism is an equivalence relation and isomorphic groups are isoclinic.

(iii) The map $\varphi\colon G' \to H'$ is uniquely determined by an isoclinism.

(iv) For $G \approx H$, it holds that $G' \cong H'$ and

$$G/\mathrm{Z}(G)G' \cong (G/\mathrm{Z}(G))/(G/\mathrm{Z}(G))' \cong (H/\mathrm{Z}(H))/(H/\mathrm{Z}(H))' \cong H/\mathrm{Z}(H)H'.$$

Therefore, $|G|$ and $|H|$ differ only by the so-called *branching factor*

$$\mathrm{Z}(G)G'/G' \cong \mathrm{Z}(G)/(\mathrm{Z}(G) \cap G').$$

For $x \in G' \cap \mathrm{Z}(G)$, it holds that $\varphi(x)\mathrm{Z}(H) = \psi(x\mathrm{Z}(G)) = 1$, d. h. $\varphi(x) \in \mathrm{Z}(H)$. The restriction of φ thus yields an isomorphism $G' \cap \mathrm{Z}(G) \cong H' \cap \mathrm{Z}(H)$.

(v) For $G \approx H$, by assumption $\varphi(G') = H'$ holds. Let us assume $\varphi(G^{[k]}) = H^{[k]}$ for some $k \geq 2$. For $y \in G^{[k]}$, it holds that $\tilde{y} \in \varphi(y)\mathrm{Z}(H) \subseteq H^{[k]}\mathrm{Z}(H)$. For $x \in G$, it follows that $\varphi([x,y]) = [\tilde{x},\tilde{y}] \in H^{[k+1]}$. This shows $\varphi(G^{[k+1]}) \subseteq H^{[k+1]}$. For reasons of symmetry, equality must hold. In particular, $G^{[k]} \cong H^{[k]}$ for all $k \geq 2$. Furthermore, $G/\mathrm{Z}_k(G) \cong H/\mathrm{Z}_k(H)$ holds for $k \geq 1$. In particular, G and H have the same nilpotency class (if nilpotent) and derived length (if solvable).

(vi) From $G/\mathrm{Z}(G) \cong H/\mathrm{Z}(H)$ and $G' \cong H'$, it does not follow that $G \approx H$ (see Example 5.48).

Example 5.42.

(i) All abelian groups are isoclinic to each other. For every abelian group A, it holds more generally that $G \approx G \times A$.

(ii) Let $\widehat{G}_1$ and $\widehat{G}_2$ be maximal Schur extensions of G. As in the proof of Theorem 5.27, let $G = F/N$ and $\widehat{G}_i = F/K_i$ for a free group F. For $L/[F,N] = \mathrm{Z}(F/[F,N])$ the map

$$\psi\colon \widehat{G}_1/\mathrm{Z}(\widehat{G}_1) \cong F/L \cong \widehat{G}_2/\mathrm{Z}(\widehat{G}_2), \qquad xK_1\mathrm{Z}(\widehat{G}_1) \mapsto xK_2\mathrm{Z}(\widehat{G}_2)$$

is a canonical isomorphism. Since the isomorphism $\widehat{G}_1' \cong F'/[F,N] \cong \widehat{G}_2'$ is also canonical, ψ is an isoclinism. Conversely, not every group isoclinic to $\widehat{G}_1$ (of the same order) is a (maximal) Schur extension of G. For example, by Example 5.26, Q_8 is a maximal Schur extension of itself, but $Q_8 \approx D_8$.

(iii) Let $H \leq G$ with $G = H\mathrm{Z}(G)$. Then $\mathrm{Z}(H) = H \cap \mathrm{Z}(G)$, $H' = G'$ and the map $H/\mathrm{Z}(H) \to G/\mathrm{Z}(G)$, $h\mathrm{Z}(H) \mapsto h\mathrm{Z}(G)$ is an isoclinism. Thus $G \approx H$ holds.

(iv) Let G be finite. If $\mathrm{Z}(G) \not\subseteq \Phi(G)$, then there exists a maximal subgroup $M < G$ with $G = M\mathrm{Z}(G)$. By (iii), $G \approx M$ holds. Repeating the argument with M, one eventually obtains a subgroup $H \leq G$ with $G \approx H$ and $\mathrm{Z}(H) \leq \Phi(H)$.

(v) Let $N \trianglelefteq G$ with $N \cap G' = 1$. Because of $[G,N] \leq N \cap G' = 1$, we have $N \leq \mathrm{Z}(G)$. Obviously $G' \to G'N/N \cong (G/N)'$, $x \mapsto xN$ is an isomorphism. For $xN \in \mathrm{Z}(G/N)$, it holds that $[x,G] \leq G' \cap N = 1$ and $x \in \mathrm{Z}(G)$. This shows $\mathrm{Z}(G/N) = \mathrm{Z}(G)/N$. Therefore $G/\mathrm{Z}(G) \cong (G/N)/\mathrm{Z}(G/N)$, $x\mathrm{Z}(G) \to xN\mathrm{Z}(G/N)$ is an isoclinism. It follows that $G \approx G/N$.

(vi) If H is a maximal Schur extension of $G/\mathrm{Z}(G)$, then $H/\mathrm{Z}(H) \cong G/\mathrm{Z}(G)$ holds by Theorem 5.36. In general, G and H are not isoclinic (consider $G = A_4$ and $H = \mathrm{SL}(2,3)$). The following theorem shows that G is nevertheless isoclinic to a Schur extension of $G/\mathrm{Z}(G)$.

Definition 5.43. G is called a *stem group*, if $\mathrm{Z}(G) \leq G'$.

Theorem 5.44 (Hall)**.** *Every group is isoclinic to a stem group.*

Proof. Let $\{g_i : i \in I\}$ be a generating system of G and $A = \bigoplus_{i\in I}\langle a_i\rangle \cong \bigoplus_{i\in I} C_\infty$. We consider

$$H := \langle (g_i, a_i) : i \in I\rangle \leq G \times A.$$

Because of $1 \times A \leq \mathrm{Z}(G \times A)$, we have $HA = G \times A$ and $G \approx G \times A \approx H$ according to Example 5.42. Because of

$$[(g_i,a_i),(g_j,a_j)] = ([g_i,g_j],1)$$

we have $H' = G' \times 1$. In H/H', the cosets of (g_i,a_i) generate a free abelian group, d. h. $H/H' \cong A$. Therefore, $\overline{Z} := \mathrm{Z}(H)/(\mathrm{Z}(H) \cap H') \cong \mathrm{Z}(H)H'/H'$ is also a free abelian group.[12] By the universal

[12] See appendix in the Algebra notes

property of free (abelian) groups, there exists a homomorphism $\alpha\colon \overline{Z} \to \mathrm{Z}(H)$ with $\mathrm{Z}(H) = \alpha(\overline{Z}) \oplus (\mathrm{Z}(H) \cap H')$. For $K := \alpha(\overline{Z})$, it holds that $H \approx H/K$, since $K \cap H' = 1$ (Example 5.42). Because of

$$\mathrm{Z}(H/K) = \mathrm{Z}(H)/K = (\mathrm{Z}(H) \cap H')K/K \leq H'K/K = (H/K)'$$

H/K is a stem group isoclinic to G. □

Remark 5.45. The stem groups in an equivalence class of isoclinic groups are exactly the representatives of minimal order, because the branch factor is trivial.

Definition 5.46. For $n \in \mathbb{N}$, let $k_n(G)$ be the number of conjugacy classes of G of length n.

Theorem 5.47. *For finite groups $G \approx H$ and $n \in \mathbb{N}$, it holds that $k_n(G)|\mathrm{Z}(H)| = k_n(H)|\mathrm{Z}(G)|$.*

Proof. Let $\psi\colon G/\mathrm{Z}(G) \to H/\mathrm{Z}(H)$, $x\mathrm{Z}(G) \mapsto \tilde{x}\mathrm{Z}(H)$ be an isoclinism. For $g, x \in G$, it holds that

$$g \in \mathrm{C}_G(x) \iff [g,x] = 1 \iff \varphi([g,x]) = 1 \iff [\tilde{g},\tilde{x}] = 1 \iff \tilde{g} \in \mathrm{C}_H(\tilde{x}).$$

This shows

$$|^G x| = |G : \mathrm{C}_G(x)| = \frac{|G/\mathrm{Z}(G)|}{|\mathrm{C}_G(x)/\mathrm{Z}(G)|} = \frac{|H/\mathrm{Z}(H)|}{|\mathrm{C}_H(\tilde{x})/\mathrm{Z}(H)|} = |H : \mathrm{C}_H(\tilde{x})| = |^H \tilde{x}|.$$

All elements in $x\mathrm{Z}(G)$ lie in conjugacy classes of the same length. For $n \in \mathbb{N}$, it follows that

$$|\mathrm{Z}(H)|k_n(G)n = |\mathrm{Z}(H)| \left| \bigcup_{\substack{x \in G \\ |^G x| = n}} x\mathrm{Z}(G) \right| = |\mathrm{Z}(H)| \sum_{\substack{x\mathrm{Z}(G) \in G/\mathrm{Z}(G) \\ |^G x| = n}} |\mathrm{Z}(G)| = |\mathrm{Z}(G)| \sum_{\substack{\tilde{x}\mathrm{Z}(H) \in H/\mathrm{Z}(H) \\ |^H \tilde{x}| = n}} |\mathrm{Z}(H)| = |\mathrm{Z}(G)|k_n(H)n. \quad \square$$

Example 5.48. The following code provides non-isoclinic groups G and H of order 64 with $G' \cong H'$ and $G/\mathrm{Z}(G) \cong H/\mathrm{Z}(H)$:

```
G:=SmallGroup(64,128);;
H:=SmallGroup(64,149);;
IdGroup(DerivedSubgroup(G))=IdGroup(DerivedSubgroup(H));
IdGroup(G/Center(G))=IdGroup(H/Center(H));
Number(ConjugacyClasses(G),K->Size(K)=2)=Number(ConjugacyClasses(H),K->Size(K)=2);
```

6 Extensions of the alternating groups

Remark 6.1. From group theory we know that the alternating groups A_n are simple for $n \geq 5$. In this chapter, we determine extensions of A_n and with A_n.

Lemma 6.2. *For $n \geq 4$, $\mathrm{C}_{S_n}(A_n) = 1$ holds. In particular, $\mathrm{Z}(S_n) = \mathrm{Z}(A_n) = 1$.*

Proof. Let $\sigma \in \mathrm{C}_{S_n}(A_n)$ and $\tau := (a,b,c) \in A_n$ be a 3-cycle. From $\sigma\tau = \tau\sigma$ it follows that ${}^\sigma\{a,b,c\} = \{a,b,c\}$. Now $\{a\}$ is the intersection of all 3-element sets $\Delta \subseteq \{1,\ldots,n\}$ that contain a (note: $n \geq 4$). This shows $\sigma(a) = a$ for $a = 1,\ldots,n$. □

Remark 6.3. For every homomorphism $\omega\colon H \to \mathrm{Out}(A_n)$ ($n \geq 4$), there is thus exactly one extension of A_n by H for the pairing ω according to Corollay 4.28. We must therefore determine $\mathrm{Out}(A_n)$.

Lemma 6.4 (GT-Exercise 39)**.** *For $n \geq 3$, $A_n = \langle (1,2,3), \ldots, (1,2,n) \rangle$.*

Proof. We argue by induction on n. The case $n = 3$ is clear. Now let $n \geq 4$. It suffices to show that $A_n = \langle A_{n-1}, (1,2,n) \rangle =: H$ holds. Suppose indirectly that $\sigma \in A_n \setminus H$. Then there exists a $k \neq n$ with ${}^{\sigma}k = n$. Choose $\tau \in A_{n-1}$ with ${}^{\tau}1 = k$. Then $\sigma\tau(1,2,n) \in A_{n-1}$ and we obtain the contradiction $\sigma \in H$. □

Lemma 6.5. *Let Ω be a set and $G \leq \mathrm{Sym}(\Omega)$ with exactly one minimal normal subgroup. Then there exists an orbit $\Delta \subseteq \Omega$ such that G acts faithfully on Δ.*

Proof. Let $\Delta_1, \ldots, \Delta_s$ be the orbits of G. Let $N \trianglelefteq G$ be the unique minimal normal subgroup and $x \in N \setminus \{1\}$. Then there exists an $1 \leq i \leq s$ such that x acts non-trivially on Δ_i. For the kernel K of the restricted action $G \to \mathrm{Sym}(\Delta_i)$, it thus holds that $K \cap N = 1$. This shows $K = 1$ and G acts faithfully on Δ_i. □

Remark 6.6.

(i) According to GT-Theorem 6.21 and GT-Theorem 6.39, the assumption of Lemma 6.5 holds for all symmetric and alternating groups.

(ii) For a k-cycle $(a_1, \ldots, a_k) \in S_n$ and an arbitrary permutation $\sigma \in S_n$, it holds that

$$\sigma(a_1, \ldots, a_k)\sigma^{-1} = (\sigma(a_1), \ldots, \sigma(a_k)).$$

This shows that all k-cycles in S_n are conjugate. By decomposing into disjoint cycles, one obtains that any two permutations $\sigma, \tau \in S_n$ are conjugate if and only if they have the same cycle type. For $\sigma \in A_n$, ${}^{S_n}\sigma \subseteq A_n$ is a union of conjugacy classes of A_n. Because of

$$|A_n : \mathrm{C}_{A_n}(\sigma)| = |A_n : A_n \cap \mathrm{C}_{S_n}(\sigma)| = |A_n\mathrm{C}_{S_n}(\sigma) : \mathrm{C}_{S_n}(\sigma)|$$

it holds that ${}^{S_n}\sigma = {}^{A_n}\sigma$ if and only if $\mathrm{C}_{S_n}(\sigma) \not\subseteq A_n$. Let $\sigma = \sigma_1 \ldots \sigma_l$ with disjoint cycles σ_i of length λ_i (including cycles of length one). If λ_i is even, then $\sigma_i \in \mathrm{C}_{S_n}(\sigma) \setminus A_n$. Now let us assume that $\lambda_1, \ldots, \lambda_l$ are odd. Suppose there exist $i \neq j$ with $\lambda_i = \lambda_j$. For $\sigma_i = (a_1, \ldots, a_{\lambda_i})$ and $\sigma_j = (b_1, \ldots, b_{\lambda_i})$, then $(a_1, b_1) \ldots (a_{\lambda_i}, b_{\lambda_i}) \in \mathrm{C}_{S_n}(\sigma) \setminus A_n$.

In the case ${}^{S_n}\sigma \neq {}^{A_n}\sigma$, the $\lambda_1, \ldots, \lambda_l$ must therefore be odd and pairwise distinct. One can show that the converse also holds.

Lemma 6.7. *Let $n \geq 4$ and $A_{n-1} \cong H \leq A_n$. Then $H = \mathrm{Alt}(\{1, \ldots, n\} \setminus \{i\})$ for some $i \in \{1, \ldots, n\}$ or $n = 6$.*

Proof. Wlog. let $n \geq 5$. According to Lemma 6.5, H acts faithfully on an orbit $\Delta \subseteq \{1, \ldots, n\}$. Because of $|H| = (n-1)!/2$, it follows that $|\Delta| \geq n - 1$. We can therefore assume that H acts transitively on $\{1, \ldots, n\}$. In particular, $n \mid |H| \mid (n-1)!$. Hence we may assume $n \geq 8$.

Let $f\colon A_{n-1} \to H$ be an isomorphism and let $\sigma \in A_{n-1}$ be a 3-cycle. Obviously, A_{n-4} is isomorphic to a subgroup of $\mathrm{C}_{A_{n-1}}(\sigma)$. In particular, $\mathrm{C}_H(f(\sigma))$ also possesses a subgroup $C \cong A_{n-4}$. Let $f(\sigma)$ be the disjoint product of k many 3-cycles. Obviously, C permutes the orbits of $f(\sigma)$ (including trivial

orbits). The kernel of this action is a 3-group and thus trivial. There is therefore a monomorphism $F\colon C \to S_{n-2k}$. Because of $|C| = (n-4)!/2$, it follows that $k \leq 2$. In the case $k = 2$, F is transitive and one obtains the contradiction $n = 3k = 6$. Thus $f(\sigma)$ is also a 3-cycle.

Let $f((1,2,3)) = (\alpha,\beta,\gamma)$ and $f((1,2,4)) = (\delta,\epsilon,\varphi)$. In the case $\{\alpha,\beta,\gamma\} \cap \{\delta,\epsilon,\varphi\} = \varnothing$,

$$A_4 \cong \langle (1,2,3),(1,2,4)\rangle \cong \langle f((1,2,3)), f((1,2,4))\rangle$$

would be abelian. In the case $|\{\alpha,\beta,\gamma\} \cap \{\delta,\epsilon,\varphi\}| = 1$, $\langle f((1,2,3)), f((1,2,4))\rangle$ would act transitively on the 5-element set $\{\alpha,\beta,\gamma\} \cup \{\delta,\epsilon,\varphi\}$. Thus $|\{\alpha,\beta,\gamma\} \cap \{\delta,\epsilon,\varphi\}| = 2$ and $\langle f((1,2,3)), f((1,2,4))\rangle$ can move only four digits. By induction on k, one sees that $\langle f((1,2,3)),\dots,f((1,2,k))\rangle$ can move at most k digits. Thus $H = f(A_{n-1}) = \langle f((1,2,3)),\dots,f((1,2,n-1))\rangle$ (Lemma 6.4) is intransitive. But this was already excluded. □

Theorem 6.8 (Hölder)**.** *It holds that* $\mathrm{Aut}(A_6) \cong S_6 \rtimes C_2$ *and* $\mathrm{Aut}(A_n) \cong S_n$ *for* $4 \leq n \neq 6$.

Proof. Obviously S_n acts by conjugation on A_n with kernel $\mathrm{C}_{S_n}(A_n) = 1$ (Lemma 6.2). This yields a monomorphism $\Psi\colon S_n \to \mathrm{Aut}(A_n)$. Let $H_i := \mathrm{Alt}(\{1,\dots,n\} \setminus \{i\})$ for $i = 1,\dots,n$. First, let $n \neq 6$. According to Lemma 6.7, $\mathrm{Aut}(A_n)$ then acts on $\{H_1,\dots,H_n\}$. This yields a homomorphism $\Gamma\colon \mathrm{Aut}(A_n) \to S_n$. Let $f \in \mathrm{Ker}(\Gamma)$ and $\sigma \in A_n$. Then

$$H_{f(\sigma)i} = f(\sigma)H_i f(\sigma)^{-1} = f(\sigma)f(H_i)f(\sigma)^{-1} = f(\sigma H_i \sigma^{-1}) = f(H_{\sigma i}) = H_{\sigma i}$$

and $f(\sigma) = \sigma$. Thus Γ is injective. Therefore Ψ and Γ are even isomorphisms. Now let $n = 6$.

Step 1: $\Psi(S_6) < \mathrm{Aut}(A_6)$.
Obviously A_5 acts faithfully and transitively on $\mathrm{Syl}_5(A_5)$ by conjugation. This yields a monomorphism $f\colon A_5 \to \mathrm{Sym}(\mathrm{Syl}_5(A_5)) \cong S_6$. In the case $f(A_5) \not\subseteq A_6$, we would have $1 \neq f(A_5) \cap A_6 \lhd f(A_5)$ in contradiction to the simplicity of $f(A_5) \cong A_5$. Thus $f(A_5) \leq A_6$. Since f is transitive, $f(A_5) \neq H_i$ for $i = 1,\dots,6$. The action of A_6 on the cosets $A_6/f(A_5)$ yields a monomorphism $\varphi\colon A_6 \to S_6$. As before, $\varphi(A_6) = A_6$, i. e. $\varphi \in \mathrm{Aut}(A_6)$. Since $f(A_5)$ is the stabilizer of the trivial coset, $\varphi(f(A_5)) = H_i$ must hold for some $i \in \{1,\dots,6\}$. In particular, $\varphi \notin \Psi(S_6)$.

Step 2: $|\mathrm{Out}(A_6)| = 4$.
Let $\varphi,\psi \in \mathrm{Aut}(A_6) \setminus \Psi(S_6)$. Obviously, every automorphism maps conjugacy classes to conjugacy classes. According to Remark 6.6, the set of 3-cycles is a conjugacy class C of A_6. If $\varphi(C) = C$, then one sees as in the proof of Lemma 6.7 that φ permutes the subgroups H_i. But then $\varphi \in \Psi(S_6)$ would hold. Thus $\varphi(C) = \psi(C)$ must be the conjugacy class of elements of cycle type $(3,3)$, because these are the only other elements of order 3. It follows that $(\varphi\psi)(C) = C$ and $\varphi\psi \in \Psi(S_6)$. This implies the claim.

Step 3: $\mathrm{Aut}(A_6) \cong S_6 \rtimes C_2$.
As a subgroup of index 2, $\Psi(S_6) \trianglelefteq \mathrm{Aut}(A_6)$. It is therefore sufficient to find an element $\varphi \in \mathrm{Aut}(A_6) \setminus \Psi(S_6)$ of order 2. First, let $\varphi \in \mathrm{Aut}(A_6) \setminus \Psi(S_6)$ be arbitrary and let $x := (1,2,3,4,5) \in A_6$. Then $\varphi(x)$ is also a 5-cycle and therefore conjugate to x in S_6 (Remark 6.6). Thus, if one replaces φ by a suitable element from the coset $\varphi\Psi(S_6)$, one can assume $\varphi(x) = x$. Thus there exists a $y \in \mathrm{C}_{S_6}(x)$ with $\varphi^2 = \Psi(y)$. Since S_6 has exactly $6 \cdot 4!$ cycles of length 5, $|\mathrm{C}_{S_6}(x)| \leq 5$ and thus $y \in \mathrm{C}_{S_6}(x) = \langle x\rangle$. This shows $|\langle\varphi\rangle| \in \{2,10\}$. Thus $\varphi^5 \in \mathrm{Aut}(A_6) \setminus \Psi(S_6)$ has order 2. □

Theorem 6.9. *For* $4 \leq n \neq 6$*, every extension with* A_n *splits.*

Proof. According to Theorem 6.8, $\mathrm{Inn}(A_n) \cong A_n$ has a complement in $\mathrm{Aut}(A_n) \cong S_n$. The claim follows from Lemma 6.2 and Corollay 4.28. □

Remark 6.10. According to Theorem 6.8, $\mathrm{Out}(A_6) \cong C_2^2$. The next theorem shows that Theorem 6.9 is false for $n = 6$.

Theorem 6.11. *The extension* $\mathrm{Aut}(A_6)$ *of* $\mathrm{Inn}(A_6) \cong A_6$ *does not split.*

Proof. Assume $A \leq \mathrm{Aut}(A_6)$ is a complement of $\mathrm{Inn}(A_6)$. Let $\Psi(S_6) \leq \mathrm{Aut}(A_6)$ be as in the proof of Theorem 6.8. Then $|\Psi(S_6) \cap A| = 2$. Let $\sigma \in S_6$ be an involution with $\alpha := \Psi(\sigma) \in A$. Because of $\mathrm{Inn}(A_6) \cap A = 1$, σ has cycle type (2) or $(2,2,2)$. Wlog. let $\sigma \in \{(1,2), (1,2)(3,4)(5,6)\}$. Let $\beta \in A \setminus \Psi(S_6)$. As in the proof of Theorem 6.8, β interchanges the 3-cycles with the permutations of cycle type $(3,3)$. Because of $|A| = 4$, A is abelian.

Case 1: $\sigma = (1,2)$.
It holds that

$$\beta((3,4,5)) = (\beta\alpha)((3,4,5)) = (\alpha\beta)((3,4,5)) = (1,2)\beta((3,4,5))(1,2),$$

but $(1,2)$ cannot centralize any permutation of cycle type $(3,3)$. Contradiction.

Case 2: $\sigma = (1,2)(3,4)(5,6)$.
Obviously, $(1,3,5)(2,4,6)$ is centralized by σ. The same calculation as in Case 1 with $(1,3,5)(2,4,6)$ instead of $(3,4,5)$ shows that σ centralizes a 3-cycle. This is also impossible. □

Remark 6.12. The exceptional behavior of $\mathrm{Aut}(A_6)$ can be explained by the (likewise exceptional) isomorphism $A_6 \cong \mathrm{PSL}(2,9)$ (Exercise 18).

```
G:=AlternatingGroup(6);;
A:=AutomorphismGroup(G);;
sub:=SubgroupsOfIndexTwo(A);;
List(sub,IdGroup);
IdGroup(SymmetricGroup(6));
IdGroup(PGL(2,9));
IdGroup(MathieuGroup(10));
```

Lemma 6.13. *For* $n \in \mathbb{N}$*, it holds that* $|M(S_n)| \leq 2$.

Proof. We use the presentation

$$S_n = \langle x_1, \ldots, x_{n-1} \mid x_i^2 = (x_i x_{i+1})^3 = [x_i, x_j] = 1 \text{ für } i < j-1 \rangle$$

from Theorem 2.18. Let F and N be as in Theorem 5.24. Let $\overline{F} := F/[F,N]$ and $\overline{x_i} := x_i[F,N] \in \overline{F}$ for $i = 1, \ldots, n-1$. Let $a_i := \overline{x_i}^2$, $b_i := (\overline{x_i x_{i+1}})^3$ and $c_{ij} := \overline{[x_i, x_j]}$ be elements in $\overline{N}$. Because of $\overline{N} \leq \mathrm{Z}(\overline{F})$, these are generators of $\overline{N}$ (one does not need the normal closure). Since one can realize x_i by the transposition $(i, i+1)$, there exist $g \in F$, $y, z \in N$ with $\overline{g} x_i \overline{g}^{-1} = x_1 y$ and $\overline{g} x_j \overline{g}^{-1} = x_3 y$ (note: $i < j-1$). Because of $\overline{N} \leq \mathrm{Z}(\overline{F})$ it follows that

$$c_{ij} = \overline{g} c_{ij} \overline{g}^{-1} = [\overline{g x_i g^{-1}}, \overline{g x_j g^{-1}}] = [\overline{x_1 y}, \overline{x_3 z}] = \overline{[x_1, x_3]} = c_{13}.$$

Furthermore,

$$c_{13}^2 a_3 = (c_{13}\overline{x_3})^2 = (\overline{x_1 x_3 x_1^{-1}})^2 = a_3$$

and thus $c_{13}^2 = 1$. A similar calculation shows

$$b_i^2 a_i^{-1} a_{i+1}^{-2} = (b_i(\overline{x_{i+1}x_ix_{i+1}})^{-1})^2 = (\overline{x_ix_{i+1}x_i})^2 = a_i^2 a_{i+1}$$

and it follows that $b_i^2 = (a_i a_{i+1})^3$. We set $d_1 := a_1$ and $d_{i+1} := b_i(a_i a_{i+1})^{-1}$. Then $d_{i+1}^2 = a_i a_{i+1}$ and

$$\overline{N} = \langle d_1, \ldots, d_{n-1}, c_{13} \rangle$$

with $c_{13}^2 = 1$. According to Theorem 5.24, the free part of $\overline{N}$ has rank $n-1$. Therefore $d_1, \ldots, d_{n-1}$ must have infinite order. The torsion part is thus $\langle c_{13} \rangle$. The claim now follows from Theorem 5.24. □

Lemma 6.14. *For $n \in \mathbb{N} \setminus \{6, 7\}$ it holds that $|M(A_n)| \leq 2$ and $|M(A_6)|, |M(A_7)| \leq 6$.*

Proof. The cases $n \leq 7$ can be handled with GAP or GT-Corollary 11.19. So let $n \geq 8$ and

$$A_n \cong \langle x_1, \ldots, x_{n-2} \mid x_1^3 = x_2^2 = \ldots = x_{n-2}^2 = (x_i x_{i+1})^3 = (x_i x_j)^2 = 1 \text{ for } |j-i| > 1 \rangle$$

as in Theorem 2.19. We use the notation from Lemma 6.13 with $a_1 := \overline{x_1}^3$, $a_i := \overline{x_i}^2$ $(i \geq 2)$, $b_i := (\overline{x_i x_{i+1}})^3$, $c_i := (\overline{x_1 x_i})^2$ $(i \geq 3)$ and $c_{ij} := [\overline{x_i}, \overline{x_j}]$ for $2 \leq i < j-1$. As in Lemma 6.13, it then follows that $c := c_{24} = c_{ij}$, $c^2 = 1$ and $b_i^2 = (a_i a_{i+1})^3$ for $i \geq 2$. For $z_{i+1} := b_i(a_i a_{i+1})^{-1}$, it thus holds that $\langle b_i, a_i a_{i+1} \rangle = \langle z_{i+1} \rangle$ for $i = 2, \ldots, n-3$. For $i \geq 3$, we have

$$a_1 = (\overline{x_i^{-1} x_1 x_i})^3 = (a_i^{-1} \overline{x_1}^{-1} c_i)^3 = a_1^{-1}(a_i^{-1} c_i)^3$$

and $\boxed{(c_i a_i^{-1})^3 = a_1^2.}$ We set $z_1 := a_1(c_3 a_3^{-1})^{-1}$ and obtain $\langle a_1, c_3 a_3^{-1} \rangle = \langle z_1 \rangle$.

From $c_4 = \overline{x_1}^{-1}(\overline{x_1 x_4})^2 \overline{x_1} = (\overline{x_4 x_1})^2$ and $c = a_2 a_4 [\overline{x_2}^{-1}, \overline{x_4}^{-1}] a_2^{-1} a_4^{-1} = [\overline{x_2}^{-1}, \overline{x_4}^{-1}]$, it follows that

$$\overline{x_4 x_1 x_2 x_4^{-1}} = c_4 \overline{x_1^{-1} x_4^{-1} x_2 x_4} a_4^{-1} = c_4 \overline{x_1^{-1} x_2} c a_4^{-1} = \overline{x_2 (x_1 x_2)^{-1} x_2^{-1}} a_2 c_4 c a_4^{-1}.$$

The third power yields

$$b_1 = b_1^{-1} a_2^3 c_4^3 c a_4^{-3}.$$

Due to $(c_4 a_4^{-1})^3 = a_1^2$, one obtains $b_1^2 = a_1^2 a_2^3 c$. With $z_2 := b_1 a_1^{-1} a_2^{-1} c$, it holds that $z_2^2 = a_2 c$ and $z_2^3 = a_1^{-1} b_1$.

For $i \geq 5$, we have

$$\overline{x_3 x_1 x_i x_3^{-1}} = c_3 \overline{x_1^{-1} x_3^{-1} x_i x_3} a_3^{-1} = c_3 \overline{x_1^{-1} x_i} c_{3i} a_3^{-1}$$

and $c_i = (c_3 a_3^{-1})^2 (\overline{x_1^{-1} x_i})^2$ after squaring. Because $\overline{x_1^{-1} x_i} = a_i \overline{x_i}^{-1} (\overline{x_1 x_i})^{-1} \overline{x_i}$, it follows that $(c_i a_i^{-1})^2 = (c_3 a_3^{-1})^2$. On the other hand, $(c_i a_i^{-1})^3 = a_1^2 = (c_3 a_3^{-1})^3$ also holds. This shows $c_i a_i^{-1} = c_3 a_3^{-1}$ for $i \geq 5$. The same calculation with index 4 instead of 3 yields $c_4 a_4^{-1} = c_6 a_6^{-1}$. Therefore, $\langle z_1 \rangle = \langle a_1, c_i a_i^{-1} \rangle$ for $i = 3, \ldots, n-2$.

Overall, we have

$$\overline{N} = \langle a_i, b_i, c_i, c \rangle = \langle z_1, \ldots, z_{n-2}, c \rangle$$

and it follows that $|M(A_n)| \leq 2$. □

Theorem 6.15. *It holds that*

$$M(S_n) = \begin{cases} 1 & \text{if } n \leq 3, \\ C_2 & \text{if } n \geq 4. \end{cases}$$

Proof. Wlog. let $n \geq 4$. According to Lemma 6.13, it suffices to construct a proper Schur extension of S_n. Let

$$\widehat{S}_n := \langle x_1, \ldots, x_{n-1}, z \mid z^2 = 1,\ x_i^2 = (x_i x_{i+1})^3 = [x_i, x_j] = z \text{ for } i < j-1 \rangle.$$

By von Dyck, there exists an epimorphism $\widehat{S}_n \to S_n$ with kernel $\langle z \rangle$. Because of $x_i^2 = [x_i, x_j] = z$, we have $z \in \mathrm{Z}(\widehat{S}_n) \cap \widehat{S}_n'$. However, $z = 1$ could hold. We define matrices in $\mathrm{GL}(2^n, \mathbb{C})$ as iterated Kronecker products of 2×2 matrices:

$$A := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad C := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$M_{2k-1} := C^{\otimes(n-k)} \otimes A \otimes 1_2^{\otimes(k-1)}, \qquad M_{2k} := C^{\otimes(n-k)} \otimes B \otimes 1_2^{\otimes(k-1)}$$

for $k = 1, \ldots, n$. Let additionally $M_0 := 0_{2^n}$. Because of $A^2 = B^2 = -C^2 = [A, B] = [A, C] = [B, C] = -1_2$, it holds that $M_k^2 = [M_k, M_l] = -I := -1_{2^n}$ for $k \neq l$. We now set

$$X_k := \frac{1}{\sqrt{2k}} \big(\sqrt{k+1} M_k - \sqrt{k-1} M_{k-1} \big)$$

for $k = 1, \ldots, n-1$. It follows that

$$\begin{aligned}
X_k^2 &= -\frac{1}{2k} \big((k+1)I + (k-1)I \big) = -I \qquad (\Longrightarrow X_k \in \mathrm{GL}(2^n, \mathbb{C})), \\
X_k X_l &= -X_l X_k \qquad (k < l-1), \\
X_{k+1} X_k &= \frac{1}{2\sqrt{k^2+k}} \Big(\sqrt{(k+2)(k+1)} M_{k+1} M_k - \sqrt{(k+2)(k-1)} M_{k+1} M_{k-1} \\
&\quad + \sqrt{k^2+k} I + \sqrt{k(k-1)} M_k M_{k-1} \Big) \\
&= I - X_k X_{k+1}, \\
X_k X_{k+1} X_k &= X_k (I - X_k X_{k+1}) = X_k + X_{k+1} = (I - X_k X_{k+1}) X_{k+1} = X_{k+1} X_k X_{k+1} \\
(X_k X_{k+1})^3 &= X_k X_{k+1} X_k X_{k+1} X_k X_{k+1} = (-I)^3 = -I.
\end{aligned}$$

By von Dyck, there exists an epimorphism $\Gamma \colon \widehat{S}_n \to \langle X_1, \ldots, X_{n-1} \rangle$ with $\Gamma(z) = -I$. Thus $z \neq 1$ in $\widehat{S}_n$. □

Remark 6.16. According to Theorem 5.19, S_n has at most $\gcd(|S_n / S_n'|, |M(S_n)|) = 2$ maximal Schur covers. In addition to $\widehat{S}_n$, there is also the Schur cover

$$\widetilde{S}_n := \langle x_1, \ldots, x_{n-1}, z \mid z^2 = 1, x_i^2 = (x_i x_{i+1})^3 = 1, [x_i, x_j] = z \text{ für } i < j-1 \rangle.$$

One easily shows $\widetilde{S}_4 \cong \mathrm{GL}(2,3)$. More generally, let q be an odd prime power, $\mathbb{F}_q^\times = \langle \zeta \rangle$ and $x := \left(\begin{smallmatrix} 0 & \zeta \\ -1 & 0 \end{smallmatrix} \right) \in \mathrm{GL}(2,q)$. Then x induces an outer automorphism α of $\mathrm{SL}(2,q)$ of order 2 by conjugation. It holds that $\mathrm{SL}(2,5) \rtimes \langle \alpha \rangle \cong \widetilde{S}_5$. Through a similar construction, one arrives at $\widetilde{S}_6 \cong \mathrm{SL}(2,9) \rtimes C_2$ (cf. Exercise 18). Now let $\zeta \in \mathbb{F}_{q^2}^\times$ be of order $2(q-1)$ and $x := \mathrm{diag}(\zeta, \zeta^{-1}) \in \mathrm{SL}(2, q^2)$. Then $K(q) := \mathrm{SL}(2,q)\langle x \rangle \leq \mathrm{SL}(2, q^2)$ is a non-split extension with $K(3) \cong \widehat{S}_4$ and $K(5) \cong \widehat{S}_5$ (without proof). An outer automorphism of S_6 yields an isomorphism $\widehat{S}_6 \cong \widetilde{S}_6$. For $n \neq 6$, however, $\widehat{S}_n \ncong \widetilde{S}_n$ (Exercise 21 and Remark 5.22). For larger n, the Schur covers of S_n are not "known" groups.

```
S1:=SchurCoverOfSymmetricGroup(5,3,1); #Ŝ5 as a matrix group over F3
S2:=SchurCoverOfSymmetricGroup(5,3,-1); #S̃5
IdGroup(S1); IdGroup(S2);
IsomorphismGroups(SchurCoverOfSymmetricGroup(4,3,-1),GL(2,3));
```

Theorem 6.17. *It holds that*

$$M(A_n) = \begin{cases} C_6 & \text{if } n \in \{6,7\}, \\ C_2 & \text{if } n = 4,5,8,9,\ldots. \end{cases}$$

Proof. One easily checks that $\mathrm{SL}(2,3)$ is a Schur extension of A_4 (cf. Exercise 15). Therefore $M(A_4) \cong C_2$ holds. Now let $n \geq 5$, $\widehat{A}_n := \widehat{S}_n'$ and $Z := \mathrm{Z}(\widehat{S}_n) \leq \mathrm{Z}(\widehat{A}_n)$. Then $\widehat{A}_n/Z \cong (\widehat{S}_n/Z)' \cong S_n' \cong A_n$ and $\widehat{A}_n'Z/Z = (\widehat{A}_n/Z)' \cong A_n' \cong A_n$. It follows that $|\widehat{S}_n/\widehat{A}_n'| = |\widehat{S}_n/\widehat{A}_n||\widehat{A}_n/\widehat{A}_n'| \leq 4$ and $Z \leq \widehat{S}_n' \leq \widehat{A}_n'$. Thus $\widehat{A}_n$ is a Schur extension of A_n and $2 \mid |M(n)|$ for $n \geq 5$. For $n \notin \{6,7\}$, $\widehat{A}_n$ is the universal Schur extension of A_n according to Remark 5.22.

It therefore suffices to construct Schur extensions $\widehat{A}_6$ and $\widehat{A}_7$ with $\mathrm{Z}(\widehat{A}_6) \cong \mathrm{Z}(\widehat{A}_7) \cong C_3$. We generate $\widehat{A}_6$ by monomial matrices in $\mathrm{GL}(6,4)$. Let $\mathbb{F}_4^\times = \langle\zeta\rangle$, $\overline{\zeta} := \zeta^{-1}$ and $(a_1,\ldots,a_6;\sigma) := (a_i\delta_{i\sigma(j)})_{ij} \in \mathrm{GL}(6,4)$ for $a_1,\ldots,a_6 \in \mathbb{F}_4$ and $\sigma \in S_6$. We define

$$\begin{aligned} x_1 &:= (\overline{\zeta},1,\zeta,1,\zeta,\overline{\zeta};(145)(263)), \\ x_2 &:= (1,1,1,1,\overline{\zeta},\zeta;(13)(56)), \\ x_3 &:= (1,\zeta,\overline{\zeta},\zeta,\overline{\zeta},1;(23)(45)), \\ x_4 &:= (\overline{\zeta},1,\zeta,1,\zeta,\overline{\zeta};(15)(36)). \end{aligned}$$

Let $G := \langle x_1,x_2,x_3,x_4\rangle$ (note that the given permutations generate a transitive A_5). A calculation shows

$$x_1^3 = x_2^2 = x_3^2 = x_4^2 = (x_1x_2)^3 = (x_2x_3)^3 = (x_3x_4)^3 = (x_1x_4)^2 = (x_2x_4)^2 = 1$$

and $z := (x_1x_3)^2 = \zeta 1_6 \in \mathrm{Z}(G)$. Furthermore $z = x_1x_3x_1x_3 = x_1x_3x_4x_1^{-1}x_4x_3 = [x_1,x_3x_4] \in G'$. According to Moore, G is a Schur extension of A_6 with $\mathrm{Z}(G) = \langle z\rangle \cong C_3$. We now add

$$x_5 := \begin{pmatrix} 1 & \overline{\zeta} & 1 & \overline{\zeta} & 1 & . \\ \zeta & 1 & \zeta & . & \zeta & 1 \\ 1 & \overline{\zeta} & 1 & \overline{\zeta} & . & \overline{\zeta} \\ \zeta & . & \zeta & 1 & \zeta & 1 \\ 1 & \overline{\zeta} & . & \overline{\zeta} & 1 & \overline{\zeta} \\ . & 1 & \zeta & 1 & \zeta & 1 \end{pmatrix}.$$

A further calculation yields $x_5^2 = (x_2x_5)^2 = (x_3x_5)^2 = (x_4x_5)^3 = 1$ and $(x_1x_5)^2 = z$. Therefore $\langle x_1,\ldots,x_5\rangle$ is the desired Schur extension $\widehat{A}_7$. □

Remark 6.18.

(i) The Schur extension $\widehat{A_6}$ with $|\mathrm{Z}(\widehat{A_6})| = 3$ constructed in the proof can be embedded as an imprimitive permutation group of degree 18 in $C_3 \wr A_6$. One calls $\widehat{A_6}$ the VALENTINER *group*.

(ii) Suppose the (universal) Schur extension $\widehat{A}_7$ has already been constructed. Then there exists a subgroup $Z \leq H \leq \widehat{A}_7$ with $H/Z \cong A_6$. For $P \in \mathrm{Syl}_3(H)$, we also have $P \in \mathrm{Syl}_3(\widehat{A}_7)$. According to Taunt (GT-Theorem 7.15), P is non-abelian and therefore $Z = [P,P] \leq H'$. Thus H is a Schur extension of A_6. It therefore suffices to construct $\widehat{A}_7$. We have already done this with GAP in Remark 5.18. The group can also be constructed in other ways:

```
PerfectGroup(IsPermGroup,15120); #only perfect group of this order
LoadPackage("atlasrep",false);
AtlasGroup("6.A7"); #requires internet on first access
DoubleCoverOfAlternatingGroup(10,3); #the Schur cover of A10 as a matrix group over F3 with Z(...) ≅ C2
```

(iii) Blackburn has constructed all extensions of elementary abelian groups by S_n and A_n.

7 Symplectic Groups

Remark 7.1. The most complicated simple groups are the groups of *Lie type*. These are certain matrix groups over finite fields, of which we have already constructed the projective special linear groups $\mathrm{PSL}(n,q)$ in Group Theory. In this chapter, the (projective) symplectic groups $\mathrm{PSp}(2n,q)$ are introduced, which only exist in even dimension. For $n=1$, we have $\mathrm{PSp}(2,q)=\mathrm{PSL}(2,q)$. For $n\geq 2$, one obtains "new" simple groups. The proof uses Iwasawa's Lemma. In contrast to $\mathrm{PSL}(n,q)$, we try to argue "coordinate-free" this time.

Definition 7.2. Let K be a field and V a finite-dimensional K-vector space. Let $\beta\colon V\times V\to K$, $(v,w)\mapsto[v,w]$ be a bilinear form with the following properties:

- (non-degenerate) For all $v\in V\setminus\{0\}$, there exists a $w\in V$ with $[v,w]\neq 0$.
- (alternating) For all $v\in V$, we have $[v,v]=0$.

With respect to $[.,.]$ one calls V a *symplectic space*. For $v\in V$ and $U\subseteq V$, let $v^\perp:=\{w\in V:[v,w]=0\}\leq V$ and $U^\perp:=\bigcap_{u\in U}u^\perp\leq V$.

Remark 7.3. From $[v,v]=0$ follows the *antisymmetry* $[u,v]=[u+v,u+v]-[v,u]=-[v,u]$ for $u,v\in V$. For $\operatorname{char}K\neq 2$, the terms "alternating" and "antisymmetric" are equivalent, because $[v,v]=-[v,v]=0$. For $\operatorname{char}K=2$, symmetry and antisymmetry are identical.

Theorem 7.4. *For $U\leq V$, we have $\dim V=\dim U+\dim U^\perp$ (but not necessarily $U\cap U^\perp=0$).*

Proof. We extend a basis $b_1,\ldots,b_k$ of U to a basis $b_1,\ldots,b_n$ of V. With respect to this basis, we identify V with K^n. For the Gram matrix $B=([b_i,b_j])$ of β, it then holds that $[v,w]=vBw^{\mathrm{t}}$ for $v,w\in V$. Since β is non-degenerate, B is invertible (alternating is not required for the proof). The first k rows of B form a matrix A of rank $k=\dim U$. Now $U^\perp$ is the solution space of the homogeneous system of equations $Ax=0$. From linear algebra, it follows that $\dim U^\perp=n-k=\dim V-\dim U$. □

Theorem 7.5. *There exists a basis $b_1,\ldots,b_n,c_1,\ldots,c_n$ of V with $[b_i,b_j]=0=[c_i,c_j]$ and $[b_i,c_j]=\delta_{ij}$ for $i,j=1,\ldots,n$. In particular, $\dim V=2n$ is even (cf. GT-Exercise 70).*

Proof. Induction on $\dim V$. Let $b_1\in V\setminus\{0\}$. Since β is non-degenerate, there exists $c_1\in V$ with $[b_1,c_1]\neq 0$. After scaling, $[b_1,c_1]=1$ holds. Now let $U:=\langle b_1,c_1\rangle$. Obviously $U\cap U^\perp=0$ and Theorem 7.4 shows $V=U\oplus U^\perp$. For $u\in U^\perp$, there exists a $v\in V$ with $[u,v]\neq 0$. Writing $v=v_1+v_2$ with $v_1\in U$ and $v_2\in U^\perp$, it follows that $[u,v_2]=[u,v]\neq 0$. Therefore, the restriction of β to $U^\perp$ is also non-degenerate and alternating. The claim now follows by induction. □

Definition 7.6.

- We call $(v,w)\in V^2$ with $[v,w]=1$ a *hyperbolic pair*. A basis as in Theorem 7.5 is called *symplectic*.
- We call

$$\begin{aligned}\mathrm{Sp}(V)&:=\big\{f\in\mathrm{GL}(V):[f(v),f(w)]=[v,w]\big\}\leq\mathrm{GL}(V),\\ \mathrm{PSp}(V)&:=\mathrm{Sp}(V)/\mathrm{Z}(\mathrm{Sp}(V))\end{aligned}$$

  the (*projective*) *symplectic group* of V.

Remark 7.7.

(i) The Gram matrix of β with respect to a symplectic basis is $B = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$. For $v, w \in K^{2n}$ with respect to this basis, $[v, w] = vBw^{\mathrm{t}}$ holds. This shows

$$\mathrm{Sp}(2n, K) := \{A \in \mathrm{GL}(2n, K) : A^{\mathrm{t}}BA = B\} \cong \mathrm{Sp}(V).$$

In particular, the isomorphism type of $\mathrm{Sp}(V)$ does not depend on β. For finite fields, we set $\mathrm{Sp}(2n, q) := \mathrm{Sp}(2n, \mathbb{F}_q)$.

(ii) For $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right) \in \mathrm{Sp}(2n, K)$, it holds that

$$\begin{pmatrix} A_1^{\mathrm{t}} & A_3^{\mathrm{t}} \\ A_2^{\mathrm{t}} & A_4^{\mathrm{t}} \end{pmatrix} \begin{pmatrix} A_3 & A_4 \\ -A_1 & -A_2 \end{pmatrix} = A^{\mathrm{t}}BA = B = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

thus $A_1^{\mathrm{t}}A_3 = A_3^{\mathrm{t}}A_1$, $A_2^{\mathrm{t}}A_4 = A_4^{\mathrm{t}}A_2$ and $A_1^{\mathrm{t}}A_4 = 1_n + A_3^{\mathrm{t}}A_2$. For $n = 1$, the first two equations are trivial and the third states $\det(A) = 1$. This shows $\mathrm{Sp}(2, K) = \mathrm{SL}(2, K)$. For arbitrary n, $\left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_1^{-\mathrm{t}} \end{smallmatrix}\right) \in \mathrm{Sp}(V)$ for all $A_1 \in \mathrm{GL}(n, K)$. Therefore, $\mathrm{GL}(n, K)$ is isomorphic to a subgroup of $\mathrm{Sp}(2n, K)$.

(iii) We can also arrange the Gram matrix of β in the form $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \oplus \ldots \oplus \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then it follows that $\mathrm{Sp}(2k, K) \times \mathrm{Sp}(2(n-k), K) \leq \mathrm{Sp}(2n, K)$ for $0 < k < n$.

Lemma 7.8. *It holds that* $\mathrm{Z}(\mathrm{Sp}(V)) = \mathrm{Sp}(V) \cap K^{\times}\,\mathrm{id} = \langle -\mathrm{id} \rangle$.

Proof. Because of $-\mathrm{id} \in \mathrm{Sp}(V) \cap K^{\times}\,\mathrm{id} \subseteq \mathrm{Z}(\mathrm{Sp}(V))$, we only need to show $\mathrm{Z}(\mathrm{Sp}(2n, K)) \subseteq \langle -1_2 \rangle$. For $n = 1$, $\mathrm{Z}(\mathrm{Sp}(2, K)) = \mathrm{Z}(\mathrm{SL}(2, K)) = \langle -1_2 \rangle$ (in the case $\mathrm{char}\, K = 2$, $-1_2 = 1_2$). So let $n \geq 2$. We write $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right) \in \mathrm{Z}(\mathrm{Sp}(2n, K))$ as in Remark 7.7. For $M = \left(\begin{smallmatrix} C & 0 \\ 0 & C^{-\mathrm{t}} \end{smallmatrix}\right) \in \mathrm{Sp}(2n, K)$ with $C \in \mathrm{GL}(n, K)$, it holds that

$$\begin{pmatrix} A_1C & A_2C^{-\mathrm{t}} \\ A_3C & A_4C^{-\mathrm{t}} \end{pmatrix} = AM = MA = \begin{pmatrix} CA_1 & CA_2 \\ C^{-\mathrm{t}}A_3 & C^{-\mathrm{t}}A_4 \end{pmatrix}.$$

This shows $A_1, A_4 \in \mathrm{Z}(\mathrm{GL}(n, K)) = K^{\times}1_n$. For $C = 1_n + E_{ij}$ with $i \neq j$, it follows that $E_{ij}A_2 = -A_2E_{ji}$ and $A_2 = 0$. Analogously, $A_3 = 0$ and one obtains $A_1^{\mathrm{t}} = A_4^{-1} \in \mathrm{GL}(n, K)$. Let $\lambda \in K^{\times}$ with $A_1 = \lambda 1_n$ and $A_4 = \lambda^{-1}1_n$. Because of $B = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right) \in \mathrm{Sp}(2n, K)$, it also holds that

$$\begin{pmatrix} 0 & A_1 \\ -A_4 & 0 \end{pmatrix} = AB = BA = \begin{pmatrix} 0 & A_4 \\ -A_1 & 0 \end{pmatrix}.$$

Thus $\lambda = \lambda^{-1} = \pm 1$. □

Theorem 7.9. *It holds that*

$$\boxed{\begin{aligned} |\mathrm{Sp}(2n, q)| &= q^{n^2} \prod_{k=1}^{n} (q^{2k} - 1), \\ |\mathrm{PSp}(2n, q)| &= \frac{1}{\gcd(q-1, 2)} q^{n^2} \prod_{k=1}^{n} (q^{2k} - 1). \end{aligned}}$$

Proof. According to Lemma 7.8, the second equation follows from the first. Every $f \in \mathrm{Sp}(V)$ maps a symplectic basis to a symplectic basis. Conversely, one can define exactly one $f \in \mathrm{Sp}(V)$ this way. Therefore, $|\mathrm{Sp}(V)|$ is the number of symplectic bases of V. For $b_1 \in V \setminus \{0\}$, there are $q^{2n}-1$ possibilities. The vector c_1 defines exactly one coset $c_1 + b_1^\perp$. Thus, there are $|b_1^\perp| = q^{2n-1}$ possibilities for c_1. In the case $n = 1$, it follows that $|\mathrm{Sp}(2,q)| = q(q^2-1)$ as claimed. Now let $n \geq 2$ and $U := \langle b_1, c_1 \rangle$. As in Theorem 7.5, $V = U \oplus U^\perp$ and $b_2, \ldots, b_n, c_2, \ldots, c_n$ form a symplectic basis of $U^\perp$. By induction on n, it follows that

$$|\mathrm{Sp}(2n,q)| = q^{2n-1}(q^{2n}-1)|\mathrm{Sp}(2n-2,q)| = q^{2n-1+(n-1)^2} \prod_{k=1}^{n}(q^{2k}-1) = q^{n^2} \prod_{k=1}^{n}(q^{2k}-1). \qquad \square$$

Definition 7.10. For $\lambda \in K$ and $v \in V$, the linear map

$$t_{\lambda,v} \colon V \to V, \qquad x \mapsto x + \lambda[x,v]v$$

is called a (*symplectic*) *transvection*.

Remark 7.11. For $\lambda, \mu \in K$ and $v, x \in V$, we have

$$t_{\lambda,v}(t_{\mu,v}(x)) = t_{\lambda,v}(x + \mu[x,v]v) = x + \mu[x,v]v + \lambda[x + \mu[x,v]v, v]v = t_{\lambda+\mu,v}(x).$$

In particular, $t_{\lambda,v}$ is invertible with $t_{\lambda,v}^{-1} = t_{-\lambda,v}$. Because of

$$[t_{\lambda,v}(x), t_{\lambda,v}(y)] = [x,y] + \lambda[x,v][v,y] + \lambda[y,v][x,v] + \lambda^2[x,v][y,v][v,v] = [x,y]$$

it follows that $t_{\lambda,v} \in \mathrm{Sp}(V)$. Obviously, $v^\perp$ is the fixed point space (eigenspace for the eigenvalue 1) of $t_{\lambda,v}$. For a hyperbolic pair (v,w), we have $t_{\lambda,v}(w) = w - \lambda v \in w + v^\perp$. This shows $\det(t_{\lambda,v}) = 1$. For $g \in \mathrm{Sp}(V)$, we have

$$(g t_{\lambda,v} g^{-1})(x) = g(g^{-1}(x) + \lambda[g^{-1}(x), v]v) = x + \lambda[x, g(v)]g(v) = t_{\lambda,g(v)}(x).$$

For $\mu \in K$, we have

$$t_{\lambda,\mu v}(x) = x + \lambda[x, \mu v]\mu v = x = x + \lambda\mu^2[x,v]v = t_{\lambda\mu^2,v}(x).$$

Lemma 7.12. *The symplectic transvections generate* $\mathrm{Sp}(V)$. *In particular,* $\mathrm{Sp}(V) \leq \mathrm{SL}(V)$.

Proof. According to Theorem 7.9, it suffices to show that $S := \langle t_{\lambda,v} : \lambda \in K, v \in V \rangle$ acts transitively on the set Ω of all symplectic bases. Let $v, w \in V \setminus \{0\}$. If $\lambda := [v,w] \neq 0$, then

$$t_{\lambda^{-1}, v-w}(v) = v + \lambda^{-1}[v, v-w](v-w) = v - (v-w) = w.$$

Otherwise, choose $x \in V \setminus (v^\perp \cup w^\perp)$. One can now realize $v \mapsto x \mapsto w$ by two transvections. Thus S is transitive on $V \setminus \{0\}$.

Now let $u, v, w \in V$ with $[u,v] = [u,w] = 1$. Then $v - w \in u^\perp$. In the case $\lambda := [v,w] \neq 0$, it holds that $t_{\lambda^{-1}, v-w}(u,v) = (u,w)$. Otherwise, set $x := u + v$. Then $[u,v] = [u,x] = [u,w] = 1$ and $[v,x] \neq 0 \neq [x,w]$. One can therefore realize $(u,v) \mapsto (u,x) \mapsto (u,w)$ by two transvections. Thus S is transitive on the set of hyperbolic pairs.

Finally, let $b_1, \ldots, b_n, c_1, \ldots, c_n$ and $b_1', \ldots, b_n', c_1', \ldots, c_n'$ be symplectic bases of V. According to what has already been shown, we can assume $(b_1', c_1') = (b_1, c_1)$. Then

$$b_2, \ldots, b_n, c_2, \ldots, c_n \quad \text{and} \quad b_2', \ldots, b_n', c_2', \ldots, c_n'$$

are symplectic bases of $\langle b_1, c_1\rangle^\perp$. By induction, there exists a product of symplectic transvections on $\langle b_1, c_1\rangle^\perp$ that maps $(b_2, \ldots, b_n, c_2, \ldots, c_n)$ to $(b_2', \ldots, b_n', c_2', \ldots, c_n')$. For $x \in \langle b_1, c_1\rangle^\perp$, b_1 and c_1 are fixed by $t_{\lambda,x}$. The transvections on $\langle b_1, c_1\rangle^\perp$ can thus be extended to $V = \langle b_1, c_1\rangle^\perp \oplus \langle b_1, c_1\rangle$ by acting trivially on $\langle b_1, c_1\rangle$. Thus one can map $b_1, \ldots, b_n, c_1, \ldots, c_n$ to $b_1', \ldots, b_n', c_1', \ldots, c_n'$ using S. □

Theorem 7.13. *For every finite field K, $\mathrm{PSp}(V)$ acts faithfully and primitively on the set Ω of 1-dimensional subspaces of V.*

Proof. For $\mathrm{Sp}(2, K) = \mathrm{SL}(2, K)$, the claim is known. So let $n \geq 2$ and $q := |K|$. According to GT-Lemma 10.7, $G := \mathrm{Sp}(V) \leq \mathrm{SL}(V)$ acts on Ω with kernel $\mathrm{Sp}(V) \cap K^\times \mathrm{id} = \langle -\mathrm{id}\rangle$. Thus $\overline{G} := \mathrm{PSp}(V)$ acts faithfully on Ω. According to the proof of Theorem 7.9, $\overline{G}$ acts transitively on Ω. Let $U := Ku \in \Omega$ and G_U be the stabilizer of U in G. Let (u, u') be a hyperbolic pair. As in the proof of Lemma 7.12, $G_u \subseteq G_U$ acts transitively on $u' + u^\perp$. Since every vector in $u' + u^\perp$ spans a different subspace, G_U has an orbit of length $\geq q^{2n-1}$ on Ω. Let $v, w \in u^\perp \setminus U$. In the case $[v, w] \neq 0$, one can map (u, v) to (u, w) as in the proof of Lemma 7.12. Now let $[v, w] = 0$. Because of

$$(v^\perp)^\perp = \langle v\rangle \neq U = (u^\perp)^\perp \neq (w^\perp)^\perp$$

it follows that $v^\perp \neq u^\perp \neq w^\perp$. Let $x \in u^\perp \setminus v^\perp$ and $y \in u^\perp \setminus w^\perp$. In the case $u^\perp \subseteq v^\perp \cup w^\perp$, we have $x \in w^\perp$, $y \in v^\perp$ and one obtains the contradiction $x + y \in u^\perp \setminus (v^\perp \cup w^\perp)$. This shows $u^\perp \nsubseteq v^\perp \cup w^\perp$. So let $x \in u^\perp \setminus (v^\perp \cup w^\perp)$. One can now realize $(u, v) \mapsto (u, x) \mapsto (u, w)$ with two transvections. Thus G_u and G_U are transitive on $u^\perp \setminus U$. This yields an orbit of length $\frac{q^{2n-1}-q}{q-1} < q^{2n-1}$. Because of

$$|\Omega \setminus \{U\}| = \frac{q^{2n} - q}{q - 1} = \frac{q^{2n-1} - q}{q - 1} + q^{2n-1}$$

G_U has two orbits on $\Omega \setminus \{U\}$ with lengths q^{2n-1} and $\frac{q^{2n-1}-q}{q-1}$.

Suppose G is imprimitive on Ω with block $\Delta \ni U$. Then one of the two non-trivial orbits of G_U must lie in Δ. However, $q^{2n-1} + 1$ and $\frac{q^{2n-1}-q}{q-1} + 1 = \frac{q^{2n-1}-1}{q-1}$ are not divisors of $|\Omega| = \frac{q^{2n}-1}{q-1} = q^{2n-1} + q^{2n-2} + \ldots + 1$. This contradiction shows that G and $\overline{G}$ act primitively. □

Lemma 7.14. *For $n \geq 2$ and $(n, q) \neq (2, 2)$, $\mathrm{Sp}(2n, q)$ is perfect.*

Proof. According to Lemma 7.12, it suffices to show that the transvections are commutators. Let first $q \geq 4$. Let $v \in V \setminus \{0\}$ and $\lambda \in K^\times$. Choose $\mu \in K^\times \setminus \{\pm 1\}$ and set $\alpha := \lambda(1 - \mu^2)^{-1}$. As is well known, there exists $g \in \mathrm{Sp}(V)$ with $g(v) = \mu^2 v$. According to Remark 7.11, it holds that

$$\mathrm{Sp}(V)' \ni [t_{\alpha,v}, g] = t_{\alpha,v} g t_{-\alpha,v} g^{-1} = t_{\alpha,v} t_{-\alpha,g(v)} = t_{\alpha,v} t_{-\alpha\mu^2,v} = t_{\alpha(1-\mu^2),v} = t_{\lambda,v}.$$

Now let $q = 3$ and $b_1, \ldots, b_n, c_1, \ldots, c_n$ be a symplectic basis of V. We define $g, h \in \mathrm{Sp}(V)$ by

$$\begin{aligned}
&g(b_1) := b_1 + b_2, && g(c_1) := c_2, && g(b_2) := b_1, && g(c_2) := c_1 - c_2,\\
&h(b_1) := b_1 - c_1 + c_2, && h(c_1) := c_1, && h(b_2) := b_2 + c_1, && h(c_2) := c_2
\end{aligned}$$

and $g(b_i) = h(b_i) = b_i$ as well as $g(c_i) = h(c_i) = c_i$ for $i \geq 3$. It holds that

$$\begin{aligned}
[g, h](b_1) &= ghg^{-1}(b_1 + c_1 - c_2) = gh(b_2 + c_2) = g(b_2 + c_1 + c_2) = b_1 + c_1,\\
[g, h](c_1) &= ghg^{-1}(c_1) = gh(c_1 + c_2) = g(c_1 + c_2) = c_1,\\
[g, h](b_2) &= ghg^{-1}(b_2 - c_1) = gh(b_1 - b_2 - c_1 - c_2) = g(b_1 - b_2) = b_2,\\
[g, h](c_2) &= ghg^{-1}(c_2) = gh(c_1) = g(c_1) = c_2,
\end{aligned}$$

i. e. $t_{1,c_1} = [g,h] \in \mathrm{Sp}(V)'$ and $t_{-1,c_1} = t_{1,c_1}^{-1} \in \mathrm{Sp}(V)'$. Since c_1 is arbitrary, the claim follows.

Finally, let $q = 2$ and $n \geq 3$. This time we define

$$\begin{aligned}
&g(b_1) := b_1 + b_3, && g(c_1) := c_3, && g(b_2) := b_1, && g(c_2) := c_1 + c_3,\\
&g(b_3) := b_2, && g(c_3) := c_2, && h(b_1) := b_1 + c_2, && h(c_1) := c_1,\\
&h(b_2) := b_2 + c_1 + c_2 + c_3, && h(c_2) := c_2, && h(b_3) := b_3 + c_2 + c_3, && h(c_3) := c_3
\end{aligned}$$

and $g(b_i) = h(b_i) = b_i$ as well as $g(c_i) = h(c_i) = c_i$ for $i \geq 4$. It holds that

$$\begin{aligned}
[g,h](b_1) &= ghg^{-1}(b_1 + c_2) = gh(b_2 + c_3) = g(b_2 + c_1 + c_2) = b_1 + c_1,\\
[g,h](c_1) &= ghg^{-1}(c_1) = gh(c_1 + c_2) = g(c_1 + c_2) = c_1,\\
[g,h](b_2) &= ghg^{-1}(b_2 + c_1 + c_2 + c_3) = gh(b_3 + c_2 + c_3) = g(b_3) = b_2,\\
[g,h](c_2) &= ghg^{-1}(c_2) = gh(c_3) = g(c_3) = c_2,\\
[g,h](b_3) &= ghg^{-1}(b_3 + c_2 + c_3) = gh(b_1 + b_2 + c_1 + c_3) = g(b_1 + b_2) = b_3,\\
[g,h](c_3) &= ghg^{-1}(c_3) = gh(c_1) = g(c_1) = c_3,
\end{aligned}$$

i. e. $t_{1,c_1} = [g,h] \in \mathrm{Sp}(V)'$. This shows the claim. □

Theorem 7.15. *It holds that* $\mathrm{Sp}(4,2) \cong S_6$. *In particular,* $\mathrm{Sp}(4,2) = \mathrm{PSp}(4,2)$ *is not perfect.*

Proof. Let $V := \{x \in \mathbb{F}_2^6 : \sum_{i=1}^6 x_i = 0\} \leq \mathbb{F}_2^6$ and $U := \langle (1,\dots,1) \rangle \leq V$. The symmetric group S_6 acts faithfully on $V/U \cong \mathbb{F}_2^4$ by permutation of the coordinates. The "standard scalar product"

$$[v + U, w + U] := \sum_{i=1}^{6} v_i w_i$$

is a well-defined alternating bilinear form on V/U. Let $v + U \neq 0$, wlog. $v_1 = v_2 = 1$ and $v_3 = 0$. Then $w + U := (0,1,1,0,0,0) + U \in V/U$ with $[v + U, w + U] \neq 0$. Thus the bilinear form is non-degenerate. Finally, for $\sigma \in S_6$ it holds that $[\sigma(v) + U, \sigma(v) + U] = [v + U, w + U]$. This shows $S_6 \leq \mathrm{Sp}(4,2)$. On the other hand, according to Theorem 7.9, $|\mathrm{Sp}(4,2)| = 2^4(2^2 - 1)(2^4 - 1) = 720 = |S_6|$. □

Theorem 7.16. *For* $n \geq 2$ *and* $(n,q) \neq (2,2)$, $\mathrm{PSp}(2n,q)$ *is simple.*

Proof. According to the previous theorems, $G := \mathrm{PSp}(2n,q)$ is perfect and primitive on the set of 1-dimensional subspaces of V. Let $U := \langle u \rangle \leq V$ with $u \neq 0$ and $A := \{\pm t_{\lambda,u} : \lambda \in K\} \leq G_U$. Obviously, A is abelian and normal in G_U. Furthermore, every transvection in $\mathrm{Sp}(V)$ is conjugate to $t_{\lambda,u}$ with some $\lambda \in K$. From Lemma 7.12 it follows that $\langle gAg^{-1} : g \in G \rangle = G$. According to Iwasawa's Lemma from group theory, G is simple. □

Remark 7.17. Another family of simple groups of Lie type can be constructed using subgroups of $\mathrm{Sp}(4,q)$ with $q = 2^{2n+1}$ and $n \in \mathbb{N}$: Let e_1, e_2, e_3, e_4 be a symplectic basis on $V = \mathbb{F}_q^4$ with $[e_1, e_3] =$

$1 = [e_2, e_4]$ We define a commutative multiplication on V by

$$e_i * e_j := \begin{cases} e_1 & \text{if } \{i,j\} = \{2,3\}, \\ e_2 & \text{if } \{i,j\} = \{1,2\}, \\ e_3 & \text{if } \{i,j\} = \{1,4\}, \\ e_4 & \text{if } \{i,j\} = \{3,4\}, \\ 0 & \text{otherwise} \end{cases}$$

$$\Big(\sum_{i=1}^{4} \lambda_i e_i\Big) * \Big(\sum_{i=1}^{4} \mu_i e_i\Big) := \sum_{i,j=1}^{4} (\lambda_i \mu_j)^{2^n} e_i * e_j$$

for $\lambda_i, \mu_j \in \mathbb{F}_q$. Let $U := \{(u,v) \in V^2 : [u,v] = 0\}$ and

$$\mathrm{Sz}(q) := \{f \in \mathrm{Sp}(V) : \forall (u,v) \in U : f(u) * f(v) = u * v\} \leq \mathrm{Sp}(V).$$

One calls $\mathrm{Sz}(q)$ the *Suzuki group* over $\mathbb{F}_q$. It is simple with order

$$|\mathrm{Sz}(q)| = q^2(q^2+1)(q-1) \equiv -1 \pmod 3.$$

The Suzuki groups are the only non-abelian simple groups whose order is not divisible by 3. Because of $q^2 + 1 \equiv 16^n \cdot 4 + 1 \equiv 0 \pmod 5$, the order of every non-abelian simple group is divisible by 3 or 5 (even by 12 or $2^6 \cdot 5$ according to Feit-Thompson). The subgroup structure and the character table of $\mathrm{Sz}(q)$ are, similar to $\mathrm{PSL}(2,q)$, very clear (the Sylow subgroups for odd primes are cyclic, cf. GT-Theorem 10.13). Suzuki also discovered a sporadic group of order $448,345,497,600$.

8 Unitary Groups

Remark 8.1. After the linear and symplectic groups, we treat in this section a third family of (simple) groups of Lie type.

Definition 8.2. Let $q \neq 1$ be a prime power, $K = \mathbb{F}_{q^2}$ and V a finite-dimensional K-vector space. Let $\mathrm{Gal}(K|\mathbb{F}_q) = \langle \alpha \rangle \cong C_2$ with $\overline{x} := \alpha(x) = x^q$ for $x \in K$ (cf. complex conjugation). Let $V \times V \to K$, $(v,w) \mapsto [v,w]$ be a non-degenerate sesquilinear form, i. e. for $u,v,w \in V$ and $\lambda \in K$ it holds that

$$\begin{aligned} \forall u \neq 0\ \exists v : [u,v] &\neq 0, \\ [u + \lambda v, w] &= [u,w] + \lambda [v,w], \\ [u,v] &= \overline{[v,u]}. \end{aligned}$$

With respect to $[.,.]$ one calls V a *unitary space*.

Remark 8.3.

(i) In the following, let V always be a unitary space.

(ii) For $u,v,w \in V$ and $\lambda \in K$ it holds that

$$[u, v + \lambda w] = \overline{[v + \lambda w, u]} = \overline{[v,u]} + \overline{\lambda}\overline{[w,u]} = [u,v] + \overline{\lambda}[u,w].$$

(iii) For $K^\times = \langle\zeta\rangle$ it holds that $\zeta\overline{\zeta} = \zeta^{q+1} \in \mathbb{F}_q$. Therefore, the *norm* $N\colon K \to \mathbb{F}_q$, $x \mapsto x\overline{x}$ is surjective. Let $x \in K$ be in the kernel of the *trace* $S\colon K \to \mathbb{F}_q$, $y \mapsto y + \overline{y}$. Then $x = -x^q$ holds and it follows that $x = 0$ or $x^{q-1} = -1$. This shows $|\mathrm{Ker}(S)| \leq q$. As a linear map, S must also be surjective according to the homomorphism theorem. For $v \in V$ it holds that $[v,v] = \overline{[v,v]} \in \mathbb{F}_q$. In the case $[v,v] \neq 0$ there exists a $\lambda \in K$ with $[\lambda v, \lambda v] = 1$, i. e. v can be *normalized*. In contrast to $K = \mathbb{C}$, there are vectors $v \neq 0$ with $[v,v] = 0$ (see Lemma 8.10).

(iv) As usual, one defines $v^\perp$ and $S^\perp \leq V$ for $v \in V$ and $S \subseteq V$. As in Theorem 7.4, one shows $\dim U + \dim U^\perp = \dim V$ for all subspaces $U \leq V$. From this, the usual rules follow:

$$(U^\perp)^\perp = U, \qquad U \subseteq W \iff W^\perp \subseteq U^\perp, \qquad (U+W)^\perp = U^\perp \cap W^\perp.$$

If $V = U \oplus U^\perp$ holds, then the restriction of $[.,.]$ to U is non-degenerate. Therefore, U is a unitary space. We will use this frequently for induction on $\dim V$.

Theorem 8.4. *Every unitary space V possesses an orthonormal basis $b_1, \ldots, b_n$, i. e. $[b_i, b_j] = \delta_{ij}$ holds.*

Proof. Let $v \in V \setminus \{0\}$. Since $[.,.]$ is non-degenerate, there exists a $w \in V$ with $[v,w] \neq 0$. After scaling, we can assume $[v,w] + [w,v] \neq 0$ (for $\mathrm{char}\, K \neq 2$ one can choose $[v,w] = 1$). In the case $[v,v] = 0 = [w,w]$, it holds that $[v+w, v+w] \neq 0$. In any case, one finds a $b_1 \in V$ with $[b_1, b_1] \neq 0$. After normalization, $[b_1, b_1] = 1$. In the case $n = 1$ we are finished. So let $n \geq 2$ and $U := b_1^\perp$. Obviously $V = \langle b_1\rangle \oplus U$ holds. According to Remark 8.3, U is a unitary space of dimension $n-1$. By induction on n, we can assume that U possesses an orthonormal basis $b_2, \ldots, b_n$. Obviously $b_1, \ldots, b_n$ is now an orthonormal basis of V. □

Example 8.5. As over $\mathbb{C}$, one sees that $V = K^n$ with respect to the *standard inner product*

$$[v,w] := v_1\overline{w_1} + \ldots + v_n\overline{w_n} \qquad (v, w \in V)$$

is a unitary space. The standard basis $e_1, \ldots, e_n$ is an orthonormal basis of V. According to Theorem 8.4, we can often restrict ourselves to this special space in the following.

Definition 8.6. A linear map between unitary spaces $f\colon V \to W$ is called an *isometry*, if $[f(u), f(v)]_W = [u,v]_V$ holds for all $u, v \in V$. One defines the groups

$$\begin{aligned} \mathrm{GU}(V) &:= \big\{ f \in \mathrm{GL}(V) : \forall v, w \in V : [f(v), f(w)] = [v,w] \big\} && (\textit{general unitary group}),\\ \mathrm{SU}(V) &:= \mathrm{GU}(V) \cap \mathrm{SL}(V) && (\textit{special unitary group}),\\ \mathrm{PSU}(V) &:= \mathrm{SU}(V)/\mathrm{Z}(\mathrm{SU}(V)) && (\textit{projective special unitary group}). \end{aligned}$$

Remark 8.7. Since $[.,.]$ is non-degenerate, every isometry is injective. Let $f \in \mathrm{GL}(V)$ with matrix $A \in \mathrm{GL}(n,K)$ w.r.t. the standard basis. For $v, w \in V$ it holds that $[f(v), f(w)] = [Av, Aw] = v^{\mathrm{t}} A^{\mathrm{t}} \overline{A}\overline{w}^{\mathrm{t}}$. By substituting the standard basis one sees

$$\forall v, w \in V : [f(v), f(w)] = [v,w] \iff A^{\mathrm{t}}\overline{A} = 1_n.$$

In this way one obtains the matrix groups

$$\begin{aligned} \mathrm{GU}(n,q) &:= \{A \in \mathrm{GL}(n,q^2) : A^{\mathrm{t}}\overline{A} = 1_n\},\ ^{13}\\ \mathrm{SU}(n,q) &:= \mathrm{GU}(n,q) \cap \mathrm{SL}(n,q^2),\\ \mathrm{PSU}(n,q) &:= \mathrm{SU}(n,q)/\mathrm{Z}(\mathrm{SU}(n,q)). \end{aligned}$$

[13] Attention: GAP uses a different basis by default. One must explicitly specify the Hermitian form: `GU(n, q, IdentityMat(n) * Z(q)^0)`.

Example 8.8. For every permutation matrix $P \in \mathrm{GL}(n, q^2)$ it holds that $P^{\mathrm{t}}\overline{P} = P^{-1}P = 1_n$, i. e. $P \in \mathrm{GU}(n, q)$. Therefore $\mathrm{GU}(n, q)$ (resp. $\mathrm{SU}(n, q)$) possesses a subgroup isomorphic to S_n (resp. A_n).

Lemma 8.9. *It holds that* $\mathrm{C}_{\mathrm{GL}(V)}(\mathrm{SU}(V)) = K^\times \mathrm{id}_V$.

Proof. Wlog. let $V = K^n$ with $n \geq 2$. Let $K^\times = \langle\zeta\rangle$. Because of $\zeta^{q-1}\overline{\zeta^{q-1}} = \zeta^{q-1+q^2-q} = \zeta^{q^2-1} = 1$ it follows that $D := \mathrm{diag}(\zeta^{q-1}, \zeta^{1-q}, 1, \ldots, 1) \in \mathrm{SU}(n, q) \setminus \{1_n\}$ (also for $q = 2$). It follows that

$$\mathrm{C}_{\mathrm{GL}(V)}(\mathrm{SU}(V)) \subseteq \mathrm{C}_{\mathrm{GL}(V)}(D) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & A \end{pmatrix} : a, b \in K^\times, A \in \mathrm{GL}(n-2, K) \right\}.$$

By permutation of the coordinates one obtains that $\mathrm{C}_{\mathrm{GL}(V)}(\mathrm{SU}(V))$ consists of diagonal matrices. Because of

$$\begin{pmatrix} 0 & \cdots & 0 & (-1)^{n-1} \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \in \mathrm{SU}(n, q)$$

only scalar matrices can centralize $\mathrm{SU}(V)$. Conversely, of course $K^\times \mathrm{id}_V \leq \mathrm{Z}(\mathrm{GL}(V))$ holds. □

Lemma 8.10. *For* $\lambda \in \mathbb{F}_q$ *let* $V_\lambda := \{v \in V : [v, v] = \lambda\}$. *Then it holds that*

$$z_n := |V_0| = q^{2n-1} + (-1)^n q^{n-1}(q-1),$$
$$w_n := |V_1| = q^{n-1}(q^n - (-1)^n).$$

Proof. Wlog. let $V = K^n$ with respect to the standard scalar product.

(i) For $n = 1$, $V_0 = \{0\}$ and $z_1 = 1 = q - (q-1)$ as claimed. Let $n \geq 1$ and $v = (v', v_n) \in K^{n+1}$ with $v' \in K^n$ and $[v, v] = 0$. In the case $[v', v'] = 0$, it follows that $v_n = 0$. There are z_n possibilities for this. Now let $[v', v'] \neq 0$ and $K^\times = \langle\zeta\rangle$. Let $a \in \mathbb{Z}$ with $-[v', v'] = \zeta^{a(q+1)}$. From

$$v_n^{q+1} = v_n\overline{v_n} = [v, v] - [v', v'] = -[v', v']$$

it follows that $v_n \in \zeta^a\langle\zeta^{q-1}\rangle$. There exist exactly $|\langle\zeta^{q-1}\rangle| = q + 1$ elements with this property. This shows

$$z_{n+1} = z_n + (q^{2n} - z_n)(q+1) = q^{2n+1} + q^{2n} - qz_n.$$

The claim follows by induction.

(ii) For $\lambda \neq 0$, $V_\lambda \to V_{\lambda\zeta^{q+1}}$, $v \mapsto \zeta v$ is a bijection. With (i) it follows that

$$w_n = |V_1| = \frac{q^{2n} - z_n}{q-1} = q^{2n-1} - (-1)^n q^{n-1} = q^{n-1}(q^n - (-1)^n).$$ □

Theorem 8.11. *It holds that*

$$\boxed{\begin{aligned} |\mathrm{GU}(n, q)| &= q^{n(n-1)/2} \prod_{k=1}^{n} (q^k - (-1)^k), \\ |\mathrm{SU}(n, q)| &= q^{n(n-1)/2} \prod_{k=2}^{n} (q^k - (-1)^k), \\ |\mathrm{PSU}(n, q)| &= \frac{q^{n(n-1)/2}}{\gcd(n, q+1)} \prod_{k=2}^{n} (q^k - (-1)^k). \end{aligned}}$$

Proof. $A \in \mathrm{GU}(n,q)$ holds if and only if the rows of A form an orthonormal basis of V with respect to $[.,.]$. For the first row a_1 of A, there are exactly w_n possibilities according to Lemma 8.10. The second row a_2 lies in $U := a_1^\perp$. Because of $V = \langle a_1 \rangle \oplus U$, U is a unitary space of dimension $n-1$. Therefore, there are w_{n-1} possibilities for a_2 etc. This shows

$$|\mathrm{GU}(n,q)| = w_1 \dots w_n = \prod_{k=1}^{n} q^{k-1}(q^k - (-1)^k) = q^{n(n-1)/2} \prod_{k=1}^{n} (q^k - (-1)^k).$$

For $A \in \mathrm{GU}(n,q)$, $\det(A)^{q+1} = \det(A)\overline{\det(A)} = \det(A^{\mathrm{t}}\overline{A}) = 1$ holds. Because of $\mathrm{diag}(\zeta^{q-1}, 1, \dots, 1) \in \mathrm{GU}(n,q)$, $\det\colon \mathrm{GU}(n,q) \to \langle \zeta^{q-1} \rangle$ is surjective with kernel $\mathrm{SU}(n,q)$. From this, the formula for $|\mathrm{SU}(n,q)|$ follows. According to Lemma 8.9,

$$\mathrm{Z}(\mathrm{SU}(n,q)) = \mathrm{SU}(n,q) \cap K^\times 1_n = \mathrm{SU}(n,q) \cap \langle \zeta^{q-1} \rangle.$$

For $\zeta^{a(q-1)} 1_n \in \mathrm{Z}(\mathrm{SU}(n,q))$ with $0 \le a \le q$, $\zeta^{a(q-1)n} = 1$ holds. Thus $a \equiv 0 \pmod{(q+1)/\gcd(n,q+1)}$ and $|\mathrm{Z}(\mathrm{SU}(n,q))| = \gcd(n, q+1)$. From this, the formula for $|\mathrm{PSU}(n,q)|$ follows. □

Lemma 8.12. *It holds that* $\mathrm{SU}(2,q) \cong \mathrm{SL}(2,q)$ *and* $\mathrm{PSU}(2,q) \cong \mathrm{PSL}(2,q)$.

Proof. We first construct a basis $\{v, v\} \subseteq V_0$ of $V = K^2$ with $[u,v] = -[v,u]$. For this, let $\mu \in K \setminus \{1\}$ with $\mu^{q+1} = \mu\overline{\mu} = -1$. For $2 \nmid q$ we set $u := (1,\mu)$, $v := (\mu, 1)$ and for $2 \mid q$ let $u := (1,1)$, $v := (1,\mu)$. In both cases u and v are linearly independent with $[u,u] = [v,v] = 0$. By Remark 8.3 it follows that $[u,v] \neq 0$. After normalization of v one obtains $[u,v] = 1$. For $2 \mid q$ it is then $[u,v] = \overline{[v,u]} = [v,u] = -[v,u]$. For $2 \nmid q$ there exists $\tau \in K$ with $\tau^{q-1} = -1$, d. h. $\overline{\tau} = -\tau$. By replacing u with τu, we obtain $[u,v] = -[v,u]$ as desired.

Let $A \in \mathrm{SU}(2,q)$ and $Au = au + cv$, $Av = bu + dv$ with $a,b,c,d \in K$. Then

$$\begin{aligned} a\overline{c} - \overline{a}c &= \frac{[Au, Au]}{[u,v]} = 0 = \frac{[Av, Av]}{[u,v]} = b\overline{d} - \overline{b}d, \\ a\overline{d} - c\overline{b} &= \frac{[Au, Av]}{[u,v]} = 1 = d\overline{a} - b\overline{c}, \\ ad - bc &= \det A = 1. \end{aligned}$$

It follows that

$$\begin{aligned} \overline{c} + bc\overline{c} &= (1+bc)\overline{c} = ad\overline{c} = d\overline{a}c = (1+b\overline{c})c = c + bc\overline{c}, \\ b + b\overline{b}c &= b(1+c\overline{b}) = ba\overline{d} = ad\overline{b} = (1+bc)\overline{b} = \overline{b} + b\overline{b}c \end{aligned}$$

and $b, c \in \mathbb{F}_q$. In the case $b \neq 0$ it follows that $d = \overline{d} \in \mathbb{F}_q$. Otherwise $a \neq 0$ and $ad = 1 = a\overline{d}$. Again $d \in \mathbb{F}_q$ holds. Analogously one obtains $a \in \mathbb{F}_q$. The basis transformation $\{(1,0),(0,1) \mapsto \{u,v\}$ thus yields a monomorphism $\varphi\colon \mathrm{SU}(2,q) \to \mathrm{SL}(2,q)$. By Theorem 8.11 we have $|\mathrm{SU}(2,q)| = q(q^2-1) = |\mathrm{SL}(2,q)|$. Thus φ is an isomorphism. □

Example 8.13. Obviously $\mathrm{GU}(1,q) \cong C_{q+1}$ and $\mathrm{SU}(1,q) = 1$. By Lemma 8.12 we have $\mathrm{PSU}(2,2) \cong \mathrm{SU}(2,2) \cong \mathrm{SL}(2,2) \cong S_3$ and $\mathrm{PSU}(2,3) \cong \mathrm{PSL}(2,3) \cong A_4$. By Theorem 8.11 we have

$$|\mathrm{SU}(3,2)| = 2^3(2^2-1)(2^3+1) = 8 \cdot 27$$

and $|\mathrm{PSU}(3,2)| = 72$. In particular, $\mathrm{PSU}(3,2)$ is solvable. More precisely, $\mathrm{PSU}(3,2) \cong M_9 \cong C_3^2 \rtimes Q_8$ (Exercise 34). The following GAP code shows $\mathrm{PSU}(4,2) \cong \mathrm{PSp}(4,3)$:

```
G:=PSU(4,2);
H:=PSp(4,3);
IsomorphismGroups(G,H);
```

Remark 8.14. As in the proof of Lemma 8.12, one shows that every unitary space V of dimension ≥ 2 possesses a hyperbolic pair (u, v), i. e. $u, v \in V_0$ and $[u, v] = 1$ (Definition 7.6).

Theorem 8.15 (Witt)**.** *Let V be a unitary space, $U \leq V$ and $\sigma\colon U \to V$ an isometry. Then there exists a $\tau \in \mathrm{GU}(V)$ with $\tau_{|U} = \sigma$.*

Proof. Induction on $\dim U$. Wlog. let $U \neq 0$. Let $W \leq U$ with $\dim(U/W) = 1$. By induction, there exists a $\tau \in \mathrm{GU}(V)$ with $\tau_{|W} = \sigma_{|W}$. By replacing σ with $\tau_{|U}^{-1}\sigma$, we can assume $\sigma_{|W} = \mathrm{id}_W$. Wlog. let $\sigma \neq \mathrm{id}_V$. Then $R := (\sigma - \mathrm{id}_U)(U) \leq V$ is 1-dimensional. For $u, v \in U$, it holds that

$$[\sigma(u), \sigma(v) - v] = [\sigma(u), \sigma(v)] - [\sigma(u), v] = [u, v] - [\sigma(u), v] = [u - \sigma(u), v]. \tag{8.1}$$

This shows $W \subseteq R^\perp$.

Case 1: $U \nsubseteq R^\perp$.
Let $R^\perp = W \oplus Y$. For dimension reasons, $V = U + R^\perp = U + W + Y = U \oplus Y$ holds. For $u \in U$ and $y \in Y$, we have $[\sigma(u) - u, y] = 0$, i. e. $[\sigma(u), y] = [u, y]$. From this it follows easily that the map $V \to V$, $u + y \mapsto \sigma(u) + y$ is a unitary extension of σ.

Case 2: $U \subseteq R^\perp$.
From (8.1) it follows that $\sigma(U) \subseteq R^\perp$. Assume $\sigma(U) \neq U$. Then $W = \sigma(W) = U \cap \sigma(U)$. Choose $u \in U \setminus W$ and $v \in \sigma(U) \setminus W$. For $U_0 := \langle u + v \rangle$ it holds that

$$U + \sigma(U) = U \oplus U_0 = \sigma(U) \oplus U_0.$$

Let $R^\perp = (U + \sigma(U)) \oplus Y$ and $S := U_0 + Y$. Then $R^\perp = U \oplus S = \sigma(U) \oplus S$ holds. In the case $\sigma(U) = U$, one can choose an arbitrary complement S of U in $R^\perp$. In both cases, (8.1) shows that

$$R^\perp \to R^\perp, \qquad u + s \mapsto \sigma(u) + s$$

for $u \in U$ and $s \in S$ is an isometry. We can therefore assume $U = R^\perp = \sigma(U)$. Let $R = \langle x \rangle \subseteq U$. Then $[x, x] = 0$ holds. Let $v \in V \setminus R^\perp$. Then $T := \langle x, v \rangle$ is a unitary space. We can replace v by a suitable element in T such that (x, v) is a hyperbolic pair (Remark 8.14). Because $T^\perp \subseteq x^\perp = R^\perp = U$, we have

$$U = \langle x \rangle \oplus T^\perp = \langle \sigma(x) \rangle \oplus \sigma(T^\perp).$$

It follows that $\dim(\langle v \rangle + \sigma(T^\perp)) = \dim(V) - 1$. Choose $w \in V$ with $\langle v \rangle + \sigma(T^\perp) = w^\perp$. Because $\langle v \rangle + \langle \sigma(x) \rangle + \sigma(T^\perp) = \langle v \rangle + U = V$, we have $\sigma(x) \notin w^\perp$. From this it follows that $w \notin U$, because $U = \sigma(U) = \sigma(x^\perp) = \sigma(x)^\perp$. Since $\langle \sigma(x), w \rangle$ is a unitary space, we can modify w such that $(\sigma(x), w)$ is a hyperbolic pair. Subsequently, $\sigma(T^\perp) \subseteq \langle \sigma(x), w \rangle^\perp$ still holds.

We define the linear map $\tau\colon V \to V$, $u + \lambda v \mapsto \sigma(u) + \lambda w$ for $u \in U$ and $\lambda \in K$. Because

$$[x, v] = 1 = [\sigma(x), w], \qquad [y, v] = 0 = [\sigma(y), w], \qquad [v, v] = 0 = [w, w]$$

for all $y \in T^\perp$, $\tau \in \mathrm{GU}(V)$ is an extension of σ. □

Lemma 8.16. *Let V be a unitary space and $v, w \in V$ linearly independent with $[v, v] = [w, w] = 1$. Then $V = \langle v, w \rangle \oplus \langle v, w \rangle^\perp$ holds if and only if $[v, w][w, v] \neq 1$.*

Proof. Let $\lambda := [v,w]$. Let $a,b \in K$ with $u := av + bw \in \langle v,w\rangle^\perp$. Then $0 = [u,v] = a + b\overline{\lambda}$ and $0 = [u,w] = a\lambda + b$. It follows $a = -b\overline{\lambda} = a\lambda\overline{\lambda}$. This shows $u = 0$ or $\lambda\overline{\lambda} = 1$. Conversely, if $\lambda\overline{\lambda} = 1$ holds, then $v - \lambda w \in \langle v,w\rangle^\perp$. □

Definition 8.17. Let $v \in V_0$ and $\lambda \in K$ with $\lambda + \overline{\lambda} = 0$. The linear map

$$t_{\lambda,v}\colon V \to V, \qquad x \mapsto x + \lambda[x,v]v$$

is called a (*unitary*) *transvection*.

Remark 8.18. For $v,x \in V$ and $\lambda,\mu \in K$ we have

$$t_{\lambda,v}(t_{\mu,v}(x)) = t_{\lambda,v}(x + \mu[x,v]v) = x + \mu[x,v]v + \lambda[x + \mu[x,v]v, v]v = t_{\lambda+\mu,v}(x).$$

In particular, $t_{\lambda,v}$ is invertible with $t_{\lambda,v}^{-1} = t_{-\lambda,v}$. For $p := \operatorname{char} K$ we also have $t_{\lambda,v}^p = t_{p\lambda,v} = t_{0,v} = \mathrm{id}_V$. From $\lambda + \overline{\lambda} = 0$ it follows that

$$[t_{\lambda,v}(x), t_{\lambda,v}(y)] = [x,y] + \lambda[x,v][v,y] + \overline{\lambda}\overline{[y,v]}[x,v] = [x,y]$$

and $t_{\lambda,v} \in \mathrm{GU}(V)$. As in the proof of Theorem 8.11, $\det(t_{\lambda,v})^{q+1} = 1$. On the other hand, $\det(t_{\lambda,v})^p = \det(t_{\lambda,v}^p) = \det(\mathrm{id}_V) = 1$. This shows $t_{\lambda,v} \in \mathrm{SU}(V)$. For $g \in \mathrm{GU}(V)$ we have $gt_{\lambda,v}g^{-1} = t_{\lambda,g(v)}$ as in Remark 7.11. For $\mu \in K^\times$ we have

$$t_{\lambda,\mu v}(x) = x + \lambda[x,\mu v]\mu v = x + \lambda\mu\overline{\mu}[x,v]v = t_{v,\lambda\mu\overline{\mu}}(x)$$

with $\lambda\mu\overline{\mu} + \overline{\lambda}\overline{\mu}\mu = 0$.

Lemma 8.19. *Let $n \geq 2$ and $(n,q) \neq (3,2)$. Then $\mathrm{SU}(n,q)$ is generated by all transvections.*

Proof. Let $V := K^n$.
Case 1: $n = 2$.
According to the proof of Lemma 8.12, there exists a basis $\{v,w\}$ of V such that $\mathrm{SU}(V)$ consists of the matrices in $\mathrm{SL}(2,q)$. According to GT-Lemma 10.8, $\mathrm{SL}(2,q)$ is generated by the elementary matrices of the form $1_2 + \lambda E_{ij}$ with $\lambda \in \mathbb{F}_q$ and $i \neq j$. It suffices to show that these matrices correspond to unitary transvections. Wlog. let $(i,j) = (1,2)$. Let $\mu := [w,v] = -[v,w] \in K$ (see proof of Lemma 8.12). It holds that $\mu^{-1} = -\overline{\mu}^{-1} = -\mu^{-q} = -\overline{\mu^{-1}}$ and $\lambda\mu^{-1} + \overline{\lambda\mu^{-1}} = \lambda(\mu^{-1} + \overline{\mu^{-1}}) = 0$. Therefore $t := t_{\lambda\mu^{-1},v}$ is a transvection with

$$t(v) = v, \qquad t(w) = w + \lambda\mu^{-1}[w,v]v = w + \lambda v.$$

Thus t corresponds to the matrix $1_2 + \lambda E_{12}$.

For the induction on n, we additionally need that $\mathrm{SU}(V)$ acts transitively on V_1. For $u,v \in V_1$, there exists by Witt an $\alpha \in \mathrm{GU}(V)$ with $\alpha(u) = v$. Let $d := \det(\alpha)$ and $u^\perp = \langle w\rangle$. We define $\beta \in \mathrm{GU}(V)$ by $\beta(u) = u$ and $\beta(w) = d^{-1}w$. Then $\alpha\beta \in \mathrm{SU}(V)$ with $\alpha\beta(u) = v$.

Case 2: $n \geq 3$.
Let $e_1,\ldots,e_n$ be the standard basis of V. Let $\alpha \in \mathrm{SU}(V)$. Then $v := \alpha(e_1)$ is a normalized vector.

Step 1: There exists a product τ of transvections with $\tau(e_1) = v$.
If e_1 and v are linearly dependent, then by the first part of the proof there exists a product τ' of transvections in $\langle e_1,e_2\rangle$ with $\tau'(e_1) = v$. By Lemma 8.16, τ' can be extended to a product of transvections on V. We can therefore assume that e_1 and v are linearly independent.

Now assume that $U := \langle e_1, v\rangle^\perp$ contains a vector u with $[u,u] \neq 0$. After normalization, $[u,u] = 1$. By Lemma 8.16, there exist products of transvections $\tau' \in \mathrm{SU}(\langle e_1, u\rangle)$ and $\tau'' \in \mathrm{SU}(\langle u, v\rangle)$ with $\tau'(e_1) = u$ and $\tau''(u) = v$. Again, τ' and τ'' can be extended to V. Then $\tau = \tau''\tau'$ has the desired property.

We may therefore assume $[u,u] = 0$ for all $u \in U$. Because $e_1 \in U^\perp \setminus U$, it follows that $U \subsetneq U^\perp$. From $n - 2 = \dim U < \dim U^\perp = 2$ it follows that $n = 3$. By assumption $q \geq 3$. If $v_i\overline{v_i} = [v, e_i][e_i, v] \neq 1$ for some $i \in \{1,2,3\}$, then as before one can map e_1 to e_i and e_i to v by means of transvections. We can thus assume $v_i\overline{v_i} = 1$ for $i = 1,2,3$. Because $1 = [v,v] = 1+1+1$, q is now even. There exist $\lambda, \mu \in \mathbb{F}_q^\times$ with $\lambda^2 + \mu^2 = (\lambda+\mu)^2 = 1$. For $u := (0, \lambda, \mu) \in V_1$ it holds that

$$[u,v][v,u] = (\lambda\overline{v_2} + \mu\overline{v_3})(\lambda v_2 + \mu v_3) = \lambda^2 + \mu^2 + \lambda\mu(v_2\overline{v_3} + \overline{v_2}v_3) = 1 + \lambda\mu(v_2\overline{v_3} + \overline{v_2}v_3).$$

Suppose $v_2\overline{v_3} + \overline{v_2}v_3 = 0$. Because $2 \mid q$, it follows that $v_2^2 = v_2^2 v_3\overline{v_3} = v_2\overline{v_2}v_3^2 = v_3^2$ and $v_2 = v_3$. Since we can map v to $w := \alpha(e_2)$ by means of transvections, we may also transfer the conditions on v to w. Thus $w_2 = w_3$ also holds. But now $[v,w] = v_1\overline{w_1} \neq 0$ would hold. This contradiction shows $[u,v][v,u] \neq 1$. One can therefore map e_1 to u and u to v by means of transvections.

Step 2: α is a product of transvections.
For $\beta := \tau^{-1}\alpha \in \mathrm{SU}(V)$ it holds that $\beta(e_1) = e_1$. Therefore β acts on $U := \langle e_2, \ldots, e_n\rangle = e_1^\perp$. By induction on n, β_U is a product of transvections, which can be extended to V. Therefore α is also a product of transvections. □

Example 8.20. Every transvection in $\mathrm{SU}(3,2)$ has order $q = 2$ according to Remark 8.18. According to Exercise 34, the transvections in $\mathrm{SU}(3,2)$ generate a proper subgroup of order 54.

Lemma 8.21. *For $n \geq 2$, $\mathrm{PSU}(n,q)$ acts faithfully and primitively on $\Omega := \big\{\langle v\rangle : v \in V_0 \setminus \{0\}\big\}$.*

Proof. Let $V := K^n$. Obviously, $\mathrm{SU}(V)$ acts on Ω via ${}^A\langle v\rangle = \langle Av\rangle$. According to Lemma 8.9, $\mathrm{Z}(\mathrm{SU}(V))$ lies in the kernel of the action. Conversely, let A be in the kernel. As in the proof of Lemma 8.12, one constructs $u, v \in V_0$ with $[u,v] = -[v,u]$. Let $Au = \lambda_1 u$ and $Av = \lambda_2 v$. Because of $u + v \in V_0$, it holds that

$$\lambda_1 u + \lambda_2 v = A(u+v) \in \langle u+v\rangle.$$

This shows $\lambda := \lambda_1 = \lambda_2$. Let $U := \langle u, v\rangle^\perp$. Because of $V = U \oplus U^\perp$, U is a unitary space. By induction on n, there exists $\mu \in K$ with $A_{|U} = \mu\,\mathrm{id}_U$. Let $w \in V_0 \cap U$. From $u + w \in V_0$ it follows that $\lambda = \mu$ and $A = \lambda 1_n \in \mathrm{Z}(\mathrm{SU}(V))$. Thus $\mathrm{PSU}(V)$ acts faithfully on Ω.

Let $v \in V_0 \setminus \{0\}$ with wlog. $v_1 v_2 \neq 0$. As is well known, there exist $q+1$ elements $\lambda \in K$ with $\lambda^{q+1} = \lambda\overline{\lambda} = -1$. In particular, such a λ exists with $v_1 + \lambda v_2 \neq 0$. Then $u := (1, \overline{\lambda}, 0, \ldots, 0) \in V_0$ and $[v,u] \neq 0$. By scaling v, one achieves $[u,v] = -[v,u]$. Now $U := \langle u, v\rangle$ is a unitary space with $\mathrm{SU}(U) \cong \mathrm{SL}(2,q)$ (Lemma 8.12). There exists an $A \in \mathrm{SU}(U)$ with $\langle Av\rangle = \langle u\rangle$. Because of $V = U \oplus U^\perp$, one can extend A to $\mathrm{SU}(V)$ by acting trivially on $U^\perp$. Let $\mu \neq \lambda$ with $\mu\overline{\mu} = -1$. Then $w := (1, \overline{\mu}) \in V_0$ with $[u,w] = 1 + \overline{\lambda}\mu \neq 0$. Thus there exists a $B \in \mathrm{SU}(\langle u, w\rangle)$ with $\langle Bu\rangle = \langle w\rangle$. Again, B can be extended to V. This shows that $\mathrm{PSU}(V)$ acts transitively on Ω.

To prove primitivity, let first $n = 2$ and $V = \langle u, v\rangle$ with $u, v \in V_0$ and $[u,v] = -[v,u]$. Every element in $\Omega \setminus \langle v\rangle$ is generated by a vector $w := u + \lambda v \in V_0$ with $\lambda \in K$. Then

$$0 = [w,w] = \lambda[v,u] + \overline{\lambda}[u,v] = (\lambda - \overline{\lambda})[v,u],$$

i. e. $\lambda = \overline{\lambda} \in \mathbb{F}_q$. Thus Ω consists exactly of the 1-dimensional subspaces of $\mathbb{F}_q u + \mathbb{F}_q v$. By means of the isomorphism $\mathrm{SU}(V) \cong \mathrm{SL}(2,q)$, $\mathrm{SU}(V)$ even acts 2-transitively on Ω (GT-Lemma 10.7). Now let $n \geq 3$. Suppose there exists a block $\Delta \subseteq \Omega$. Let $\langle u\rangle, \langle v\rangle \in \Delta$ be distinct.

Case 1: $[u,v] \neq 0$.
Let also $w \in V_0$ with $[u,w] \neq 0$. After scaling, we can assume $[u,v] = [u,w]$. Then there exists an isometry $\alpha\colon \langle u,v\rangle \to \langle u,w\rangle$ with $\alpha(u) = u$ and $\alpha(v) = w$. According to Witt, α can be extended to $\alpha \in \mathrm{GU}(V)$. For $d := \det(\alpha)$, it holds that $d^{q+1} = d\overline{d} = 1$, i. e. $d = \zeta^{(q-1)a}$ for some $a \in \mathbb{Z}$. We define $U := \langle u,v\rangle$ and $\beta \in \mathrm{GL}(U)$ with $\beta(u) = \zeta^{-qa}u$ and $\beta(v) = \zeta^a v$. Because of

$$[\beta(u),\beta(v)] = \zeta^{-qa+qa}[u,v] = [u,v]$$

$\beta \in \mathrm{GU}(U)$ with $\det(\beta) = \zeta^{-qa+a} = d^{-1}$. Because of $V = U \oplus U^\perp$, we can extend β as $\beta + \mathrm{id}_{U^\perp}$ to $\mathrm{GU}(V)$. Then $\alpha\beta \in \mathrm{SU}(V)$ with $\alpha\beta(u) \in \langle u\rangle$ and $\alpha\beta(v) \in \langle w\rangle$. Therefore Δ contains all $\langle v\rangle$ with $[u,v] \neq 0$.

Now let $\langle w\rangle \in \Omega$ with $w \in u^\perp$. Since V is not the union of two proper subspaces, there exists an $x \in V \setminus (u^\perp \cup w^\perp)$. After scaling, let $[x,u] = 1$. Since the trace is transitive, there exists a $\lambda \in K$ with $\lambda + \overline{\lambda} = -[x,x]$ $(\in \mathbb{F}_q)$. For $x' := x + \lambda u$, it holds that

$$[x',x'] = [x,x] + (\lambda + \overline{\lambda})[u,x] = 0,$$

i. e. $x' \in V_0$. Because of $u \in w^\perp$, it holds that $x' \notin u^\perp \cup w^\perp$. From $x' \notin u^\perp$ it follows that $\langle x'\rangle \in \Delta$. From $w \notin (x')^\perp$ it follows that $\langle w\rangle \in \Delta$. This yields the contradiction $\Delta = \Omega$.

Case 2: $[u,v] = 0$.
Let $x \in u^\perp \setminus v^\perp$ with $[x,v] = 1$. Let $\lambda \in K$ with $\lambda + \overline{\lambda} = -[x,x]$. For $x' := x + \lambda v \in u^\perp \setminus v^\perp$, it holds that $[x',x'] = 0$ as in Case 1. Now $U := \langle x',v\rangle$ is a unitary space. There exists an $\alpha \in \mathrm{SU}(V)$ with $\alpha(U) = U$, $\alpha(v) \in \langle x'\rangle$ and $\alpha_{|U^\perp} = \mathrm{id}_{|U^\perp}$. Because of $u \in U^\perp$, it holds that $\langle u\rangle = \langle \alpha(u)\rangle \in \Delta \cap \alpha(\Delta) = \Delta$. This shows $\langle x'\rangle = \langle \alpha(v)\rangle \in \Delta$. From Case 1, one obtains the contradiction $\Delta = \Omega$. □

Remark 8.22. According to Lemma 8.10, $z_3 = q^5 - q^2(q-1) = (q^2-1)(q^3+1)+1$ holds. Therefore $G := \mathrm{PSU}(3,q)$ is a primitive group of degree $|\Omega| = \frac{z_3-1}{|K^\times|} = q^3+1$.

Lemma 8.23. *For $n \geq 3$ and $(n,q) \neq (3,2)$, $\mathrm{SU}(n,q)$ is perfect.*

Proof. According to Lemma 8.19, it suffices to show that every unitary transvection $t := t_{\lambda,u}$ is a product of commutators. Let $v \in V \setminus u^\perp$ such that (u,v) is a hyperbolic pair. Then $U := \langle u,v\rangle$ is a unitary space and $t_{|U^\perp} = \mathrm{id}_{U^\perp}$ because of $U^\perp \subseteq u^\perp$. According to Lemma 8.12, $t_{|U} \in \mathrm{SU}(U) \cong \mathrm{SL}(2,q)$.

First, let $q \geq 4$. According to GT-Lemma 10.9, $\mathrm{SL}(2,q)$ is perfect. Thus $t_{|U}$ is a product of commutators $[x,y] \in \mathrm{SU}(U)$. As usual, one can extend x and y to V by acting trivially on $U^\perp$. Therefore $[x,y]$ is the restriction of a commutator $c \in \mathrm{SU}(V)$ with $c_{|U^\perp} = \mathrm{id}_{U^\perp}$. Thus t is a product of commutators.

Now let $q = 3$. Like U, $U^\perp$ is also a unitary space. Therefore there exists a $w \in U^\perp$ with $[w,w] = 1$. Let $W := \langle u,v,w\rangle$. As in the first part of the proof, it suffices to show that $t_{|W} \in \mathrm{SU}(W)$ is a product of commutators. Let $\mu \in K$ with $\mu\overline{\mu} = -1$. With respect to the basis $\{u,v,w\}$, $d := \mathrm{diag}(\mu,-\mu,-\mu^{-2}) \in \mathrm{SU}(W)$. It holds that

$$t_{|W} = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} d \begin{pmatrix} 1 & \lambda/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} d^{-1} = [t_{\lambda/2,\,u}, d] \in \mathrm{SU}(W)'.$$

Finally, let $q = 2$ and $n \geq 4$. We choose a hyperbolic pair (u', v') with $u', v' \in U^\perp$. Wlog. let $V = U \oplus \langle u', v' \rangle$.[14] From $\lambda + \overline{\lambda} = 0$ it follows that $\lambda = \overline{\lambda} \in \mathbb{F}_2$. In the case $\lambda = 0$, $t = \mathrm{id}_V$ is a commutator. So let $\lambda = 1$. Let $K^\times = \langle \zeta \rangle$ and $x, y \in \mathrm{GL}(V)$ with

$$\begin{aligned} x(u) &:= u, & x(u') &:= u', & x(v) &:= v + \zeta^2 u', & x(v') &:= \zeta u + v', \\ y(u) &:= u, & y(u') &:= \zeta^2 u + u', & y(v) &:= v + \zeta v', & y(v') &:= v' \end{aligned}$$

Because of $\zeta + \zeta^2 = 1$, $x, y \in \mathrm{SU}(V)$ and $x^2 = 1$, $y^2 = 1$. For $z := [x, y]$ it holds that

$$\begin{aligned} z(u) &= u, \\ z(u') &= xy(\zeta^2 u + u') = \zeta^2 u + \zeta^2 u + u' = u', \\ z(v) &= xyx(v + \zeta v') = xy(v + \zeta^2 u' + \zeta^2 u + \zeta v') = x(v + \zeta v' + \zeta u + \zeta^2 u' + \zeta^2 u + \zeta v') \\ &= x(u + v + \zeta^2 u') = u + v, \\ z(v') &= xy(\zeta u + v') = \zeta u + \zeta u + v' = v'. \end{aligned}$$

This shows $t = z \in \mathrm{SU}(V)'$. □

Theorem 8.24. *For $n \geq 3$ and $(n, q) \neq (3, 2)$, $\mathrm{PSU}(n, q)$ is simple.*

Proof. Let $V = K^n$. We identify the elements in $\mathrm{SU}(V)$ with their cosets in $G := \mathrm{PSU}(V)$. According to Lemma 8.21, G is a primitive permutation group on $\Omega = \{\langle v \rangle : v \in V_0 \setminus \{0\}\}$. According to Lemma 8.23, G is perfect. Let $v \in V_0$ and

$$K_0 := \{\lambda \in K : \lambda + \overline{\lambda} = 0\} \cong \mathbb{F}_q$$

be the kernel of the trace. According to Remark 8.18, $A := \{t_{\lambda, v} : \lambda \in K_0\} \leq G$ is an elementary abelian p-group, where $p := \mathrm{char}\, K$. For $g \in G_{\langle v \rangle}$, there exists a $\mu \in K$ with $g(v) = \mu v$. According to Remark 8.18, it holds that

$$g t_{\lambda, v} g^{-1} = t_{\lambda,\, g(v)} = t_{\lambda,\, \mu v} = t_{\lambda \mu \overline{\mu},\, v} \in A.$$

This shows $A \trianglelefteq G_{\langle v \rangle}$. Let $w \in V_0 \setminus \{0\}$ be arbitrary. Since G acts transitively on Ω, there exists a $g \in G$ with $g(v) \in \langle w \rangle$. Therefore $A^G := \langle gAg^{-1} : g \in G \rangle$ contains all unitary transvections. From Lemma 8.19 it follows that $A^G = G$. According to Iwasawa's Lemma from group theory, G is simple. □

Example 8.25. The smallest simple group that we did not know yet is $\mathrm{SU}(3, 3) = \mathrm{PSU}(3, 3)$ with order $3^3(3^2 - 1)(3^3 + 1) = 2^5 \cdot 3^3 \cdot 7 = 6048$.

Remark 8.26. Wall has shown that a matrix $A \in \mathrm{GL}(n, q^2)$ is conjugate to an element from $\mathrm{GU}(n, q)$ if and only if A and $\overline{A}^{-1}$ are similar (i.e., conjugate in $\mathrm{GL}(n, q^2)$). Furthermore, matrices from $\mathrm{GU}(n, q)$ are conjugate in $\mathrm{GU}(n, q)$ if and only if they are similar.

Example 8.27.

(i) Let $A \in \mathrm{GL}(n, q)$ be an upper triangular matrix with ones on the main diagonal. Because of $A^{q^n} - 1_n = (A - 1_n)^{q^n} = 0$, all eigenvalues of A are equal to 1. According to the Jordan normal form, the similarity class of A is uniquely determined by $\mathrm{rk}((A - 1_n)^k)$ for $k = 1, \ldots, n$. We show $\mathrm{Ker}((A - 1_n)^k) = \mathrm{Ker}((A^{-1} - 1_n)^k)$ for $k = 1, \ldots, n$. For $k = 1$, it holds that

$$x \in \mathrm{Ker}(A - 1_n) \iff Ax = x \iff A^{-1}x = x \iff x \in \mathrm{Ker}(A^{-1} - 1_n).$$

[14] One could now use $\mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$ and Lemma 7.14.

Now let the claim be already proven for k. For $x \in \mathrm{Ker}((A-1_n)^{k+1})$, it holds inductively

$$\begin{aligned}(A-1_n)^k(A-1_n)x = 0 &\Longrightarrow (A^{-1}-1_n)^k(A-1_n)x = 0 \Longrightarrow (A-1_n)(A^{-1}-1_n)^k x = 0\\ &\Longrightarrow (A^{-1}-1_n)(A^{-1}-1_n)^k x = 0 \Longrightarrow x \in \mathrm{Ker}((A^{-1}-1_n)^{k+1}).\end{aligned}$$

Thus A and $A^{-1} = \overline{A}^{-1}$ possess the same Jordan normal form. According to Wall, A is similar to a unitary matrix.

(ii) As is well known, the upper triangular matrices with ones on the main diagonal form a p-Sylow subgroup P of $\mathrm{GL}(n,q)$, where $p \mid q$. For a p-Sylow subgroup Q of $\mathrm{GU}(n,q)$, it holds that $|Q| = |P|$ according to Theorem 8.11. Although according to (i) every element from P is conjugate to an element from Q, P and Q are in general not isomorphic. For $(n,q) = (3,2)$, $P \cong D_8$ and $Q \cong Q_8$ according to Exercise 34.

Remark 8.28.

(i) Among the so-called *classical* groups of Lie type, we are only missing the family of *orthogonal* groups. For this, one considers an n-dimensional $\mathbb{F}_q$-vector space V with a non-degenerate *quadratic form* $\rho\colon V \to K$, i. e. $\rho(\lambda v) = \lambda^2\rho(v)$ holds for $\lambda \in \mathbb{F}_q$, $v \in V$ and

$$\beta\colon V \times V \to K, \qquad (u,v) \mapsto \rho(u+v) - \rho(u) - \rho(v)$$

is a (symmetric) non-degenerate bilinear form. If q is odd, then ρ is uniquely determined by β, because

$$\rho(v) = \frac{1}{2}\Big(4\rho(v) - \rho(v) - \rho(v)\Big) = \frac{1}{2}\Big(\rho(v+v) - \rho(v) - \rho(v)\Big) = \frac{1}{2}\beta(v,v).$$

In this case, ρ is not needed and one can work with β as usual. In general, the isomorphism type of the *general orthogonal group*

$$\mathrm{GO}(n,q,\rho) := \big\{f \in \mathrm{GL}(V) : \forall v \in V : \rho(f(v)) = \rho(v)\big\}$$

depends on ρ. For the sake of simplicity, we assume that q is odd. Then there are exactly two non-equivalent bilinear forms β (Exercise 35). If n is odd, then the isomorphism type of $\mathrm{GO}(n,q,\rho)$ does not depend on ρ or β. One can then identify β with the usual "Euclidean" scalar product and define

$$\mathrm{GO}(n,q) := \mathrm{GO}(n,q,\rho) \cong \big\{A \in \mathrm{GL}(n,q) : A^{\mathrm{t}}A = 1_n\big\}$$

(for even q, $\mathrm{GO}(2n+1,q) \cong \mathrm{Sp}(2n,q)$ holds). If n is even, one defines $\mathrm{GO}^+(n,q)$ and $\mathrm{GO}^-(n,q)$ according to the choice of β. In contrast to the previous groups, the derived groups PSO, PSO^+ and PSO^- are usually not simple, but possess a simple subgroup $\mathrm{P\Omega}$ (resp. $\mathrm{P\Omega}^+$, $\mathrm{P\Omega}^-$) of index 2 (kernel of the *spinor norm*). These groups only provide "new" families of simple groups for $n \geq 7$, because

$$\begin{aligned}&\mathrm{P\Omega}(3,q) \cong \mathrm{PSL}(2,q), && \mathrm{P\Omega}^+(4,q) \cong \mathrm{PSL}(2,q)^2, && \mathrm{P\Omega}^-(4,q) \cong \mathrm{PSL}(2,q^2),\\ &\mathrm{P\Omega}(5,q) \cong \mathrm{PSp}(4,q), && \mathrm{P\Omega}^+(6,q) \cong \mathrm{PSL}(4,q), && \mathrm{P\Omega}^-(6,q) \cong \mathrm{PSU}(4,q).\end{aligned}$$

The smallest groups of this type are $\mathrm{P\Omega}^+(8,2)$, $\mathrm{P\Omega}^-(8,2)$ and $\mathrm{P\Omega}(7,3)$ with orders $174,182,400$, $197,406,720$ and $4,585,351,680$.

(ii) We construct a family of *exceptional* groups of Lie type as a subgroup of $\mathrm{GO}(7,q)$ for odd q. For this, let V be an $\mathbb{F}_q$-vector space with basis $b_0,\dots,b_7$. On $\{b_0,\dots,b_7\}$ one defines a (non-associative) multiplication $*$ with identity element $1 := b_0$ and multiplication table

$*$	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	-1	b_3	$-b_2$	b_5	$-b_4$	b_7	$-b_6$
b_2	$-b_3$	-1	b_1	b_6	b_7	$-b_4$	$-b_5$
b_3	b_2	$-b_1$	-1	b_7	$-b_6$	b_5	$-b_4$
b_4	$-b_5$	$-b_6$	$-b_7$	-1	b_1	b_2	b_3
b_5	b_4	$-b_7$	b_6	$-b_1$	-1	$-b_3$	b_2
b_6	b_7	b_4	$-b_5$	$-b_2$	b_3	-1	b_1
b_7	b_6	b_5	b_4	$-b_3$	$-b_2$	$-b_1$	-1

Note that $\langle b_1,b_2\rangle = \{\pm 1,\pm b_1,\pm b_2,\pm b_3\} \cong Q_8$. By extending $*$ distributively to V, one obtains the *octonions* $\mathbb{O} = (V,+,*)$ (also called Cayley *algebra*, although formally it is not even a ring). Now

$$\mathrm{G}_2(q) := \big\{f \in \mathrm{GL}(V) : \forall v,w \in V : f(v*w) = f(v)*f(w)\big\}$$

is a simple group. As usual, one shows that 1 is the only identity element in $\mathbb{O}$. In particular, $f(1) = 1$ holds for $f \in \mathrm{G}_2(q)$. Let $U := \langle b_1,\dots,b_7\rangle \le V$ and $i \ge 1$. Let $f(b_i) = \lambda 1 + u$ with $\lambda \in \mathbb{F}_q$ and $u \in U$. Then

$$-1 = f(-1) = f(b_i * b_i) = f(b_i) * f(b_i) = \lambda^2 1 + 2\lambda u + u * u.$$

Because of $b_i * b_j = -b_j * b_i$ for $1 \le i < j \le 7$, it holds that $u*u \in \mathbb{F}_q 1$. A comparison of coefficients shows $\lambda = 0$ or $u = 0$. In the second case, f would not be injective because of $f(b_i) = f(\lambda 1)$. Thus $f(b_i) = u \in U$ and $f(U) = U$. Clearly, $\rho(u) = -u*u = \lambda_1^2 + \ldots + \lambda_7^2$ for $u = \sum \lambda_i b_i$ defines a quadratic form on U. The corresponding bilinear form $(u,v) \mapsto \frac{1}{2}\big(\rho(u+v) - \rho(u) - \rho(v)\big)$ is the standard scalar product with respect to $b_1,\dots,b_7$, and therefore non-degenerate. Because of

$$\rho(f(u)) = -f(u) * f(u) = f(-u*u) = f(\rho(u)) = \rho(u)$$

it now holds that $f_{|U} \in \mathrm{GO}(U)$. In this way, one can view $\mathrm{G}_2(q)$ as a subgroup of $\mathrm{GO}(7,q)$. The smallest group of this type is $\mathrm{G}_2(3)$ of order $4,245,696$. For even q, $\mathrm{G}_2(q)$ can be constructed differently, where $\mathrm{G}_2(2)' \cong \mathrm{SU}(3,3)$ has index 2 in $\mathrm{G}_2(2)$.

9 Sporadic Groups

Remark 9.1. According to the classification of finite simple groups (CFSG), besides C_p, A_n and the groups of Lie type, there are 26 *sporadic* groups that do not belong to any of the families. In this chapter, we construct five sporadic groups, which are simultaneously interesting multiply transitive groups. We first choose a path as direct as possible using a "tailor-made" lemma by Witt. Subsequently, we describe a combinatorial approach with which other sporadic groups can also be constructed.

Definition 9.2. A k-transitive action $G \to \mathrm{Sym}(\Omega)$ is called *sharply* k-transitive if for two k-tuples $(\alpha_1,\dots,\alpha_k)$ and $(\beta_1,\dots,\beta_k)$ with pairwise distinct elements, there exists *exactly* one $g \in G$ with ${}^g\alpha_i = \beta_i$ for $i = 1,\dots,k$.

Remark 9.3. Every sharply k-transitive action is faithful. The sharply 1-transitive actions are exactly the regular actions. Let $G \to \mathrm{Sym}(\Omega)$ be transitive, $\omega \in \Omega$ and $k \geq 2$. As in GT-Lemma 6.33, one shows that G acts sharply k-transitively on Ω if and only if G_ω acts sharply $(k-1)$-transitively on $\Omega \setminus \{\omega\}$. By induction, one obtains that a k-transitive action of degree n is sharply k-transitive if and only if $|G| = n(n-1)\dots(n-k+1)$ holds (cf. GT-Lemma 6.34).

Example 9.4.

(i) The natural action of S_n on $\{1, \dots, n\}$ is sharply n-transitive and sharply $(n-1)$-transitive if $n \geq 2$.

(ii) The natural action of A_n is sharply $(n-2)$-transitive if $n \geq 2$ (see GT-Example 6.32).

(iii) Let $n \in \mathbb{N}$, p be a prime number and $S \leq \mathrm{GL}(n,p)$ be a Singer cycle (GT-Example 6.23). Then $\mathbb{F}_p^n \rtimes S \leq \mathrm{AGL}(n,p)$ is a sharply 2-transitive permutation group on $\mathbb{F}_p^n$.

(iv) According to GT-Exercise 49, $\mathrm{SL}(2,2^n)$ acts 3-transitively on the set Ω of all 1-dimensional subspaces of $\mathbb{F}_{2^n}^2$. Because of $|\Omega| = 2^n + 1$ and $|\mathrm{SL}(2,2^n)| = (2^{2n}-1)2^n = (2^n+1)2^n(2^n-1)$, this action is sharply 3-transitive.

Lemma 9.5. *Let $\alpha, \beta \in \Omega$, $H \leq G \leq \mathrm{Sym}(\Omega)$ and $a, x \in G$ with the following properties:*

- *$\alpha \neq \beta$, ${}^a\alpha \neq \alpha$ and ${}^x\beta \neq \beta$,*
- *$x \in H$ and $G = \langle H, a \rangle$,*
- *$aH_\beta a = H_\beta$,*
- *H acts k-transitively on $\Omega \setminus \{\alpha\}$ with $k \geq 2$,*
- *$a^2 = x^2 = (ax)^3 = 1$.*

Then G acts $(k+1)$-transitively on Ω and $G_\alpha = H$.

Proof. Because $H \subseteq G_\alpha$, G_α is k-transitive on $\Omega \setminus \{\alpha\}$. Because ${}^a\alpha \neq \alpha$, G is clearly also transitive on Ω. From GT-Lemma 6.33 it follows that G operates $(k+1)$-transitively. It remains to show: $G_\alpha \subseteq H$.

For $K := H \cup HaH$, we have $K^{-1} := \{g^{-1} : g \in K\} = K$ because $a^{-1} = a$. Let $z \in H \setminus H_\beta$. Because $k \geq 2$, H_β operates transitively on $\Omega \setminus \{\alpha, \beta\}$. Thus there exists an $h \in H_\beta$ with ${}^{hz}\beta = {}^x\beta$. It follows that $x^{-1}hz \in H_\beta$ and $z \in H_\beta x H_\beta$. This shows $H = H_\beta \cup H_\beta x H_\beta$. The relations $a^2 = x^2 = (ax)^3 = 1$ imply $axa = xax$. By assumption we obtain

$$\begin{aligned} aHa &= aH_\beta a \cup aH_\beta x H_\beta a = H_\beta \cup H_\beta axa H_\beta \\ &= H_\beta \cup H_\beta xax H_\beta \subseteq H \cup HaH = K. \end{aligned}$$

For $g, g' \in HaH$, we thus have $gg' \in HaHaH \subseteq HKH \subseteq K$. This shows $K \leq G$. Because $a \in K$, we even have $G = \langle H, a \rangle = K$. For every $g \in G \setminus H \subseteq HaH$, we thus have ${}^g\alpha \neq \alpha$. This shows the claim. □

Lemma 9.6 (Witt). *Let H be a 2-transitive permutation group on $\Delta := \{4, \dots, n\} \ni \omega$ and let $x \in H \setminus H_\omega$ be an involution. Let $a, b, c \in \mathrm{N}_{S_n}(H_\omega)$ be involutions with*

$$a = (1,\omega)(2)(3)\dots, \qquad b = (1,2)(3)(\omega)\dots, \qquad c = (2,3)(1)(\omega)\dots$$

and

$$(ax)^3 = (ba)^3 = (cb)^3 = 1, \qquad (xb)^2 = (xc)^2 = (ac)^2 = 1.$$

Then $G := \langle H, a, b, c\rangle$ *is* 5*-transitive on* $\{1, \dots, n\}$ *and* $G_1 \cap G_2 \cap G_3 = H$.

Proof. According to Lemma 9.5 with $(\alpha, \beta) = (1, \omega)$, $K := \langle H, a\rangle$ is 3-transitive on $\Delta \cup \{1\}$ and $K_1 = H$. According to GT-Theorem 6.35, H operates primitively on Δ. In particular, $H_\omega < H$ is maximal and $H = \langle H_\omega, x\rangle$. From $x^2 = (xb)^2 = b^2 = 1$ it follows that $xb = bx$. In particular, $bK_1b = bHb = \langle bH_\omega b, x\rangle = \langle H_\omega, x\rangle = H = K_1$. Another application of Lemma 9.5 with $(b, a, 2, 1)$ instead of (a, x, α, β) shows that $L := \langle K, b\rangle$ operates 4-transitively on $\Delta \cup \{1, 2\}$ with $L_2 = K$. From the relations it follows again that $ac = ca$ and $xc = cx$. Thus

$$cL_2c = cKc = \langle cHc, a\rangle = \langle cH_\omega c, x, a\rangle = \langle H_\omega, x, a\rangle = K = L_2.$$

A third application of Lemma 9.5 with $(c, b, 3, 2)$ instead of (a, x, α, β) finally yields that $G = \langle L, c\rangle$ operates 5-transitively on $\{1, \dots, n\}$ with $G_3 = L$. Thus also $G_1 \cap G_2 \cap G_3 = G_1 \cap L_2 = G_1 \cap K = K_1 = H$. □

Theorem 9.7 (Mathieu)**.** *Let*

$$\begin{aligned} a &= (1,4)(7,8)(9,11)(10,12), & b &= (1,2)(7,10)(8,11)(9,12),\\ c &= (2,3)(7,12)(8,10)(9,11), & d &= (4,5,6)(7,8,9)(10,11,12),\\ e &= (4,7,10)(5,8,11)(6,9,12), & f &= (5,7,6,10)(8,9,12,11),\\ g &= (5,8,6,12)(7,11,10,9). \end{aligned}$$

Then $M_{12} := \langle a, b, c, d, e, f, g\rangle \leq S_{12}$ *is sharply* 5*-transitive of degree* 12 *and* $M_{11} := \langle a, b, d, e, f, g\rangle$ *is sharply* 4*-transitive of degree* 11.

Proof. Since d permutes the three cycles of e, $E := \langle d, e\rangle$ is elementary abelian of order 9. Furthermore, E acts regularly on $\Delta := \{4, \dots, 12\}$. Obviously $f^2 = g^2$ is an involution and $fgf^{-1} = g^{-1}$. For $Q := \langle f, g\rangle$ it thus holds that $\langle g\rangle \trianglelefteq Q$ and $|Q : \langle g\rangle| = 2$. Thus $|Q| = 8$ (Q is a quaternion group). A calculation shows

$$\begin{aligned} fdf^{-1} &= e, & gdg^{-1} &= (4,8,12)(11,6,7)(9,10,5) = de,\\ fef^{-1} &= d^{-1}, & geg^{-1} &= (4,11,9)(8,6,10)(12,7,5) = de^{-1}. \end{aligned}$$

Thus $Q \subseteq \mathrm{N}_{S_{12}}(E)$ and $H := EQ \leq S_{12}$. For order reasons $E \cap Q = 1$ and therefore $|H| = |E||Q| = 9 \cdot 8$. Since E acts regularly on Δ, $H_4 = E_4Q = Q$. It is easy to see that Q acts transitively on $\Delta \setminus \{4\}$. Thus H is 2-transitive on Δ according to GT-Lemma 6.33. Because of $|H| = 9 \cdot 8$ and $|\Omega| = 9$, the action is even sharply 2-transitive. We now want to apply Witt's Lemma with $\omega = 4$ and

$$x := df^2d^{-1} = d(5,6)(7,10)(8,12)(9,11)d^{-1} = (4,6)(7,12)(8,11)(9,10) \in H \setminus H_4.$$

For this, we must first show $a, b, c \in \mathrm{N}_{S_{12}}(H_4) = \mathrm{N}_{S_{12}}(Q)$:

$$\begin{aligned} afa^{-1} &= g, & aga^{-1} &= a^2fa^{-2} = f,\\ bfb^{-1} &= f^{-1}, & bgb^{-1} &= (5,11,6,9)(7,12,10,8) = gf,\\ cfc^{-1} &= g^{-1}, & cgc^{-1} &= c^2f^{-1}c^{-2} = f^{-1}. \end{aligned}$$

The relations from Lemma 9.6 are verified as follows:

$$\begin{aligned} ax &= (1,4,6)(7,10,11)(8,9,12), & ba &= (1,4,2)(7,11,12)(8,10,9),\\ cb &= (1,3,2)(7,8,9)(10,12,11), & xb &= (1,2)(4,6)(7,9)(10,12),\\ xc &= (2,3)(4,6)(8,9)(10,11), & ac &= (1,4)(2,3)(7,10)(8,12). \end{aligned}$$

Thus $G := \langle H, a, b, c\rangle = M_{12}$ is 5-transitive on $\Omega = \{1, \dots, 12\}$ and $G_1 \cap G_2 \cap G_3 = H$. Since H acts sharply 2-transitive on Ω, $G_1 \cap G_2 \cap G_3 \cap G_4 \cap G_5 = H_4 \cap H_5 = 1$. This shows that G is even sharply 5-transitive. In the proof of Lemma 9.6, it resulted that $G_3 = M_{11}$. According to Remark 9.3, M_{11} is thus sharply 4-transitive of degree 11. □

Remark 9.8. In group theory, we have shown that the simple group $H = \mathrm{PSL}(3,4)$ of order

$$|H| = \frac{(4^3-1)(4^3-4)(4^3-4^2)}{(4-1)\gcd(3,4-1)} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 = 20.160$$

acts 2-transitively on the set Δ of the 21 1-dimensional subspaces of $\mathbb{F}_4^3$ (proof of GT-Lemma 10.7). To distinguish vectors from permutations, we write the elements of $\mathbb{F}_4^3$ in the form $[r,s,t]$ with $r,s,t \in \mathbb{F}_4$. Furthermore, let $\mathbb{F}_4^\times = \langle\zeta\rangle$ and $[[r,s,t]] = \mathbb{F}_4[r,s,t] \in \Delta$.

Lemma 9.9. *With the notation from Remark 9.8, the following maps are involutions in* $\mathrm{Sym}(\Delta)$*:*

$$^{\alpha}[[r,s,t]] := [[r^2+st, s^2, t^2]], \qquad ^{\beta}[[r,s,t]] := [[r^2, s^2, t^2\zeta]], \qquad ^{\gamma}[[r,s,t]] := [[r^2, s^2, t^2]]$$

for $r,s,t \in \mathbb{F}_4$.

Proof. Since the values of α, β, and γ are homogeneous polynomials of degree 2 in r,s,t, the images of $[[r,s,t]] = \mathbb{F}_4[r,s,t]$ do not depend on the choice of the representative $[r,s,t]$ (i. e., α, β, γ are well-defined). As is well known, $\mathbb{F}_4 \to \mathbb{F}_4$, $x \mapsto x^2$ is the Frobenius automorphism (of order 2). Therefore,

$$\begin{aligned} ^{\alpha^2}[[r,s,t]] &= {}^{\alpha}[[r^2+st, s^2, t^2]] = [[r+s^2t^2+s^2t^2, s, t]] = [[r,s,t]],\\ ^{\beta^2}[[r,s,t]] &= {}^{\beta}[[r^2, s^2, t^2\zeta]] = [[r, s, t\zeta^2\zeta]] = [[r,s,t]],\\ ^{\gamma^2}[[r,s,t]] &= [[r,s,t]]. \end{aligned}$$

Consequently, α, β, γ are invertible and have order 2. □

Theorem 9.10 (Mathieu)**.** *Let* $\mathrm{PSL}(3,4) \cong H \le \mathrm{Sym}(\Delta)$ *as in Remark 9.8 and* $\Omega = \Delta \,\dot\cup\, \{1,2,3\}$*. With the notation from Lemma 9.9, let* $a := (1,[[1,0,0]])\alpha$*,* $b := (1,2)\beta$ *and* $c := (2,3)\gamma$*. Then:*

(i) $M_{22} := \langle H, a\rangle$ *is 3-transitive on* $\Delta \cup \{1\}$*.*

(ii) $M_{23} := \langle H, a, b\rangle$ *is 4-transitive on* $\Omega \setminus \{3\}$*.*

(iii) $M_{24} := \langle H, a, b, c\rangle$ *is 5-transitive on* Ω*.*

Proof. We use Witt's Lemma with $\omega = [[1,0,0]] \in \Delta$ and

$$x := \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix} \mathbb{F}_4^\times \in \mathrm{PSL}(3,4).$$

According to Lemma 9.9, a, b, c, x are involutions. Clearly ${}^b\omega = {}^c\omega = \omega$. Let F be the Frobenius automorphism on H (resp. $\mathrm{SL}(3,4)$). For $h \in H$ and $[[r,s,t]] \in \Delta$ it holds that ${}^{chc}[[r,s,t]] = {}^c[h[r^2,s^2,t^2]] = [F(h)[r,s,t]]$. This shows $cHc^{-1} = H$ and $c \in \mathrm{N}_{24}(H_\omega)$. Let $y := \mathrm{diag}(1,1,\zeta) \in \mathrm{GL}(3,4)$. Then it holds that

$$ {}^{bhb}[[r,s,t]] = {}^b[h[r^2,s^2,t^2\zeta]] = [yF(h)y^{-1}[r,s,t]]. $$

Because of $yF(h)y^{-1} \in H$, we also have $b \in \S_{24}(H_\omega)$. Furthermore, ${}^{aha}\omega = {}^{ah}1 = {}^a1 = \omega$. An element $h \in H_\omega$ has the form ${}^h[[r,s,t]] = [[r+\sigma s+\tau t, \rho_1 s+\rho_2 t, \rho_3 s+\rho_4 t]]$ with $\rho_1\rho_4 + \rho_2\rho_3 = \det(h) = 1$. A calculation shows

$$ \begin{aligned} {}^{aha}[[r,s,t]] &= {}^{ah}[[r^2+st, s^2, t^2]] = {}^a[[r^2+st+\sigma s^2+\tau t^2, \rho_1 s^2+\rho_2 t^2, \rho^3 s^2+\rho_4 t^2]] \\ &= [[r+s^2t^2+\sigma^2 s+\tau^2 t+(\rho_1 s^2+\rho_2 t^2)(\rho_3 s^2+\rho_4 t^2), \rho_1^2 s+\rho_2^2 t, \rho_3^2 s+\rho_4^2 t]] \\ &= [[r+(\sigma^2+\rho_1\rho_3)s+(\tau^2+\rho_2\rho_4)t, \rho_1^2 s+\rho_2^2 t, \rho_3^2 s+\rho_4^2 t]]. \end{aligned} $$

Thus aha corresponds to the matrix

$$ \begin{pmatrix} 1 & \sigma^2+\rho_1\rho_3 & \tau^2+\rho_2\rho_4 \\ 0 & \rho_1^2 & \rho_2^2 \\ 0 & \rho_3^2 & \rho_4^2 \end{pmatrix} \in H_\omega, $$

since $\det(aha) = \rho_1^2\rho_4^2 + \rho_2^2\rho_3^2 = \det(h)^2 = 1$. Again it follows that $c \in \mathrm{N}_{S_{24}}(H_\omega)$.

Clearly $xb = bx$, $xc = cx$ and $ac = ca$ hold. This shows $(xb)^2 = (xc)^2 = (ac)^2 = 1$. It remains to show: $(ax)^3 = (ba)^3 = (cb)^3 = 1$. This results from the following calculations:

$$ \begin{aligned} {}^{cbc}[[r,s,t]] &= {}^c[[r,s,t\zeta]] = [[r^2,s^2,t^2\zeta^2]] = {}^b[[r,s,t\zeta^2]] = {}^{bcb}[[r,s,t]], \\ {}^{bab}[[r,s,t]] &= {}^b[[r+s^2t^2\zeta, s, t\zeta^2] = [[r^2+st\zeta^2, s^2, t^2\zeta^2]] \overset{1+\zeta=\zeta^2}{=} {}^a[[r+s^2t^2, s, t\zeta]] = {}^{aba}[[r,s,t]], \\ {}^{axa}1 &= {}^a[[0,1,0]] = [[0,1,0]] = {}^{xax}1, \\ {}^{axa}[[1,0,0]] &= [[1,0,0]] = {}^{xax}[[1,0,0]], \\ {}^{axa}[[0,1,0]] &= 1 = {}^{xax}[[1,0,0]], \\ {}^{axa}[[r,s,t]] &= {}^a[[s^2, r^2+st, t^2]] = [[s+r^2t^2+st^3, r+s^2t^2, t]] \\ &= \left\{ \begin{matrix} [[r^2, s^2+rt, t^2]] \text{ if } t \neq 0, \\ [[s,r,0]] \text{ if } t = 0 \neq rs \end{matrix} \right\} = {}^x[[s^2+rt, r^2, t^2]] = {}^{xax}[[r,s,t]]. \end{aligned} $$

Witt's Lemma now shows that M_{24} acts 5-transitively on Ω. From the proof of Witt's Lemma one obtains $(M_{24})_3 = M_{23}$ and $M_{22} = (M_{23})_2$. Thus the remaining statements follow. □

Definition 9.11. One calls M_{11}, M_{12}, M_{22}, M_{23} and M_{24} the *Mathieu groups* of degree 11, 12, 22, 23 and 24 respectively. From Remark 9.3 it follows that

$$ \begin{aligned} |M_{11}| &= 11 \cdot 10 \cdot 9 \cdot 8 = 7.920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11, \\ |M_{12}| &= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95.040 = 2^6 \cdot 3^3 \cdot 5 \cdot 11, \\ |M_{22}| &= 22|\mathrm{PSL}(3,4)| = 443.520 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, \\ |M_{23}| &= 23|M_{22}| = 10.200.960 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23, \\ |M_{24}| &= 24|M_{23}| = 244.823.040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23. \end{aligned} $$

Lemma 9.12. *Let $G \leq \mathrm{Sym}(\Omega)$ be 3-transitive of degree $d = |\Omega| \geq 5$. If G_ω is simple for an $\omega \in \Omega$, then G is simple or d is a power of 2.*

Proof. The proof works as for the simplicity of the alternating groups. Let $1 \neq N \trianglelefteq G$. According to GT-Theorem 6.20, N acts transitively. The Frattini argument yields $G = G_\omega N$. We can assume $G_\omega \not\leq N$ by contradiction. From the simplicity of G_ω it follows that $N_\omega = G_\omega \cap N = 1$. Thus N is a regular normal subgroup and $|N| = d \geq 5$. According to GT-Lemma 6.19, G_ω acts 2-transitively on $N \setminus \{1\}$ by conjugation. From GT-Theorem 6.36 it follows that $d = |N| = 2^k$ for a $k \in \mathbb{N}$. □

Example 9.13. Let $d = 2^n \geq 8$ and $\Omega := \mathbb{F}_2^n$. According to linear algebra, the simple group $\mathrm{GL}(n, 2)$ acts 2-transitively on $\Omega \setminus \{0\}$. Therefore, the non-simple group $\mathrm{AGL}(n, 2) = \mathbb{F}_2^n \rtimes \mathrm{GL}(n, 2)$ acts 3-transitively on Ω with stabilizer $G_0 = \mathrm{GL}(n, 2)$ (GT-Example 6.23).

Theorem 9.14. *The Mathieu groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} are simple.*

Proof (Chapman). Let first $G = M_{11}$ and $P \in \mathrm{Syl}_{11}(G)$. In S_{11} there are 10! elements of order 11, which are distributed over 9! Sylow groups. Thus $|\mathrm{N}_{S_{11}}(P)| = 11 \cdot 10$ and $|\mathrm{N}_G(P) : P|$ divides 10. According to Sylow,

$$5 \equiv 10 \cdot 9 \cdot 8 = \frac{|G|}{11} = |G : \mathrm{N}_G(P)||\mathrm{N}_G(P) : P| \equiv |\mathrm{N}_G(P) : P| \pmod{11}$$

and thus $|\mathrm{N}_G(P) : P| = 5$ and $|G : \mathrm{N}_G(P)| = 16 \cdot 9$. Now let $1 \neq N \trianglelefteq G$. Then N is transitive and all 11-Sylow groups of G lie in N. In particular, $|N : \mathrm{N}_N(P)| = 16 \cdot 9$. This shows $|G : N| \leq 5$. Let us assume $|G : N| = 5$. Then N possesses exactly $16 \cdot 9 \cdot 10$ elements of order 11. The remaining $|N|/11$ elements must then form the stabilizer N_1. In particular, $N_1 = \ldots = N_{11}$. Then N cannot act faithfully. Thus $G = N$ and G is simple.

The simplicity of M_{12} now follows from Lemma 9.12, since M_{11} is a stabilizer of M_{12}. Since $\mathrm{PSL}(3, 4)$ is simple, one can also show the simplicity of M_{22}, M_{23} and M_{24} using Lemma 9.12. □

Remark 9.15.

(i) Occasionally, the Mathieu groups M_9, M_{10}, M_{20} and $M_{21} \cong \mathrm{PSL}(3, 4)$ are defined as suitable stabilizers of the larger Mathieu groups. However, they are not sporadic simple groups. In the proof of Theorem 9.7, we constructed $M_9 \cong C_3^2 \rtimes Q_8$ as a sharply 2-transitive group of degree 9. In particular, M_9 is a Frobenius group.

(ii) One can show (elementarily) that S_k, S_{k+1}, A_{k+2}, M_{11} and M_{12} are the only sharply k-transitive permutation groups with $k \geq 4$.[15] With the CFSG, it was possible to show that S_n, A_n, M_{11}, M_{12}, M_{23} and M_{24} are the only 4-transitive permutation groups.

(iii) One can also generate the Mathieu groups with only two permutations each, but then it is difficult to determine the structure (let alone the order). In fact, Mathieu's work from 1861 met with incomprehension for a long time. For instance, Miller claimed in 1898 that M_{24} did not exist. According to the CFSG, it is known that every finite simple group can be generated by an involution and another element of prime order.

(iv) The following definition can be motivated as follows: How many lottery tickets must one buy if one wants to be guaranteed four "correct" numbers? In the best case, every combination of 4 occurs on only one ticket (whether this is possible, we will see in Example 9.19).

[15] See notes on permutation groups.

Definition 9.16. A ((t,k,v)-)*Steiner system* is a pair $S = (\Omega, \mathcal{B})$ with the following properties:

- Ω is a v-element set of "points" (vertices).
- $\mathcal{B}$ is a set of k-element subsets of Ω, which are called "blocks".[16]
- Every t-element subset of Ω lies in exactly one block of $\mathcal{B}$.

One calls

$$\mathrm{Aut}(S) := \{\sigma \in \mathrm{Sym}(\Omega) : \forall B \in \mathcal{B} : \sigma(B) \in \mathcal{B}\} \leq \mathrm{Sym}(\Omega)$$

the *automorphism group* of S.

Example 9.17.

(i) (t,k,v)-Steiner systems can obviously only exist for $t \leq k \leq v$. In the case $t = k$, $\mathcal{B}$ is the set of all k-element subsets of Ω and $\mathrm{Aut}(S) = \mathrm{Sym}(\Omega)$. In the case $k = v$, $\mathcal{B} = \{\Omega\}$ and $\mathrm{Aut}(S) = \mathrm{Sym}(\Omega)$. These Steiner systems are called *trivial*. We can therefore assume $t < k < v$.

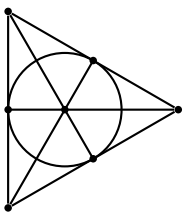
(ii) In the case $t = 1$, $\mathcal{B}$ is a partition of Ω and it follows that $k \mid v$. The number of these Steiner systems is the Stirling number[17] of the second kind $\left\{ {v \atop k} \right\}$. We see in Lemma 9.18 that there are in general strong restrictions on t, k, v.

(iii) Let q be a prime power, $n \geq 2$, $\Omega = \mathbb{F}_q^n$ and

$$\mathcal{B} = \{\mathbb{F}_q v + w : v, w \in \Omega, v \neq 0\}$$

the set of "lines" on Ω (mentally replace $\mathbb{F}_q$ with $\mathbb{R}$). Since any two distinct points lie on exactly one line, $S = (\Omega, \mathcal{B})$ is a $(2, q, q^n)$-Steiner system. S is called an *affine plane* over $\mathbb{F}_q^n$.[18]

(iv) Let $n \geq 3$, $\Omega = \{\mathbb{F}_q v : v \in \mathbb{F}_q^n \setminus \{0\}\}$ be the set of 1-dimensional subspaces of $\mathbb{F}_q^n$ and $\mathcal{B}$ the set of 2-dimensional subspaces of $\mathbb{F}_q^n$ (formally: each block consists of the 1-dimensional subspaces of a 2-dimensional space). Since any two linearly independent vectors span a 2-dimensional subspace, $S = (\Omega, \mathcal{B})$ is a $(2, q+1, \frac{q^n-1}{q-1})$-Steiner system. S is called a *projective plane* over $\mathbb{F}_q^n$. For $(q,n) = (2,3)$ one obtains the *Fano plane*:



For $(q,n) = (2,4)$ one obtains a solution to KIRKMAN's *schoolgirl problem*: 15 girls are to walk in groups of three on seven consecutive days, such that any two girls are in the same group of three only once.

Lemma 9.18. *Let $S = (\Omega, \mathcal{B})$ be a (t,k,v)-Steiner system and $0 \leq s \leq t$. Then:*

(i) The number of blocks containing a given s-element subset of Ω is

$$\gamma_s := \frac{(v-s)(v-s-1)\dots(v-t+1)}{(k-s)(k-s-1)\dots(k-t+1)}.$$

[16] This has nothing to do with the blocks of an imprimitive action.
[17] See notes for discrete mathematics.
[18] See notes for synthetic geometry.

(ii) $|\mathcal{B}| = \gamma_0 = \frac{v\gamma_1}{k}$.

(iii) (FISHER*'s inequality*) $v \leq |\mathcal{B}|$ *and* $k \leq \gamma_1$.

Proof.

(i) Every s-element subset lies in exactly $\binom{v-s}{t-s}$ t-element subsets of Ω. Every t-element subset lies by definition in exactly one block. Every block contains exactly $\binom{k-s}{t-s}$ s-element subsets. This shows

$$\gamma_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} = \frac{(v-s)(v-s-1)\dots(v-t+1)}{(k-s)(k-s-1)\dots(k-t+1)}.$$

(ii) Follows from (i).

(iii) Let $\Omega = \{\omega_1, \dots, \omega_v\}$ and $\mathcal{B} = \{B_1, \dots, B_r\}$. Let $M := (m_{ij}) \in \mathbb{Z}^{v\times r}$ be the incidence matrix of S, i.e., $m_{ij} = 1$ if $\omega_i \in B_j$ and $m_{ij} = 0$ otherwise. Then

$$MM^t = \Big(\sum_{l=1}^{r} m_{il}m_{jl}\Big)_{ij} = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_2 \\ \gamma_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma_2 \\ \gamma_2 & \cdots & \gamma_2 & \gamma_1 \end{pmatrix} = (\gamma_1 - \gamma_2)1_v + \gamma_2 J,$$

where $J = (1) \in \mathbb{Z}^{v\times v}$ is the matrix consisting only of ones. According to (i), $\gamma_1 = \frac{v-1}{k-1}\gamma_2 > \gamma_2$. The eigenvalues of J are known to be 0 with multiplicity $v-1$ and v with multiplicity 1. Therefore, $\gamma_1 - \gamma_2 > 0$ and $\gamma_1 - \gamma_2 + \gamma_2 v$ are the eigenvalues of MM^t. In particular, MM^t is invertible. This shows

$$v = \mathrm{rk}(MM^{\mathrm{t}}) \leq \mathrm{rk}(M) \leq \min\{r, v\}$$

and it follows that $v \leq r = |\mathcal{B}|$. From (ii) we obtain $k = \frac{v\gamma_1}{r} \leq \gamma_1$. □

Example 9.19. For the optimal solution of the lottery problem from Remark 9.15, a $(4, 6, 49)$-Steiner system is required. Because of $\gamma_3 = \frac{46}{3} \notin \mathbb{N}$, this does not exist. In any case, at least

$$\gamma_0 = \frac{49 \cdot 48 \cdot 47 \cdot 46}{6 \cdot 5 \cdot 4 \cdot 3} > 14,125$$

lottery tickets are required.[19]

Remark 9.20.

(i) For the (alternative) construction of the Mathieu groups, one starts with the affine plane S over $\mathbb{F}_3^2$ as a $(2, 3, 9)$-Steiner system. Now one extends S by adding three points to a $(5, 6, 12)$-Steiner system $\hat{S}$ (the details are extremely elaborate[20]). Subsequently, one defines $M_{12} := \mathrm{Aut}(\hat{S})$. The larger Mathieu groups can also be obtained in this way. In general, MENDELSOHN proved that every finite group is the automorphism group of a $(2, 3, v)$-Steiner system and a $(3, 4, v)$-Steiner system.

[19] It can be shown that at least 14,749 lottery tickets are necessary. Whether this is actually sufficient, however, is open. See `https://ljcr.dmgordon.org/show_cover.php?v=49&k=6&t=4`.

[20] See [Dixon-Mortimer, *Permutation Groups*, Springer, New York, 1996]. In the errata it is mentioned that the proof of Theorem 6.3B is incomplete.

(ii) For M_{22}, one uses a $(3,6,22)$-Steiner system $S = (\Omega, \mathcal{B})$ (it holds that $\mathrm{Aut}(S) \cong M_{22} \rtimes C_2$). According to Lemma 9.18,

$$|\mathcal{B}| = \frac{22 \cdot 21 \cdot 20}{6 \cdot 5 \cdot 4} = 77.$$

Let Γ be a graph with vertex set $\Gamma_E := \Omega \cup \mathcal{B} \cup \{\zeta\}$, so $|\Gamma_E| = 22 + 77 + 1 = 100$. Let the vertex ζ be adjacent to all $\omega \in \Omega$. Two blocks are adjacent in Γ if and only if they are disjoint. All other edges have the form $(\omega, B) \in \Omega \times \mathcal{B}$ with $\omega \in B$. According to Exercise 26, Γ is a 22-regular graph. It is called the *Higman-Sims graph*. It holds that $\mathrm{Aut}(\Gamma) = HS \rtimes C_2$, where HS is the sporadic simple *Higman-Sims group* of order $44,352,000$.

(iii) The Mathieu groups can also be constructed using methods from coding theory. The (extended binary) *Golay code* is the 12-dimensional subspace $C \leq \mathbb{F}_2^{24}$ spanned by the rows of the matrix

$$\begin{pmatrix}
1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & 1 \\
. & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & 1
\end{pmatrix}$$

One can show that

$$\mathrm{Aut}(C) := \{\sigma \in S_{24} : \forall (c_i) \in C : (c_{\sigma(i)}) \in C\}$$

is isomorphic to M_{24}. The vectors (*codewords*) with exactly eight ones in C correspond to the blocks of a $(5,8,24)$-Steiner system (the positions of the ones are the blocks). The Golay code was used for data transmission of the Voyager space probes.

(iv) The automorphism groups and Schur multipliers are

G	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}
$\mathrm{Out}(G)$	1	C_2	C_2	1	1
$M(G)$	1	C_2	C_{12}	1	1

(without proof). For all sporadic simple groups S, $|\mathrm{Out}(S)| \leq 2$ holds. For the correct calculation of $M(M_{22})$, three attempts were needed. "Worse" is only $M(M_{21}) = M(\mathrm{PSL}(3,4)) \cong C_{12} \times C_4$. In GAP, one can for example do the following:

```
G:=MathieuGroup(24);;
AllPrimitiveGroups(NrMovedPoints,24,Transitivity,5); #gives only one
PrimitiveIdentification(G); #G=PrimitiveGroup(24,1)
IsSimple(G);
H:=Stabilizer(G,24); #= M23

LoadPackage("guava",false); #package for codes
C:=ExtendedBinaryGolayCode();
Display(GeneratorMat(C));
A:=AutomorphismGroup(C);
Transitivity(A); #5-transitive
c:=Positions(Basis(C)[1],Z(2)^0); #codeword with exactly eight ones
B:=Orbit(A,c,OnSets);; #blocks of the (5,8,24)-Steiner system
Size(B); #759 = (24·23·22·21·20)/(8·7·6·5·4)
```

(v) The group M_{23} is the only sporadic group for which the *inverse Galois problem* has not yet been solved, i.e., it is not known whether there exists a number field $K \subseteq \mathbb{C}$ with $\mathrm{Gal}(K|\mathbb{Q}) \cong M_{23}$.

(vi) Closely related to the Golay code is the *Leech lattice* $L \leq \mathbb{Z}^{24}$, which consists of the integer linear combinations of the rows of the following matrix:

$$\begin{pmatrix}
8 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
2 & 2 & 2 & 2 & . & . & . & . & 2 & 2 & 2 & 2 & . & . & . & . & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & . & . & . & . & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . \\
2 & 2 & . & . & 2 & 2 & . & . & 2 & 2 & . & . & 2 & 2 & . & . & . & . & . & . & . & . & . & . \\
2 & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . & . & . & . & . & . & . & . & . \\
2 & . & . & 2 & 2 & . & . & 2 & 2 & . & . & 2 & 2 & . & . & 2 & . & . & . & . & . & . & . & . \\
4 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 4 & . & . & . & . & . & . & . \\
2 & . & 2 & . & 2 & . & . & 2 & 2 & 2 & . & . & . & . & . & . & 2 & 2 & . & . & . & . & . & . \\
2 & . & . & 2 & 2 & 2 & . & . & 2 & . & 2 & . & . & . & . & . & 2 & . & 2 & . & . & . & . & . \\
2 & 2 & . & . & 2 & . & 2 & . & 2 & . & . & 2 & . & . & . & . & 2 & . & . & 2 & . & . & . & . \\
. & 2 & 2 & 2 & 2 & . & . & . & 2 & . & . & . & 2 & . & . & . & 2 & . & . & . & 2 & . & . & . \\
. & . & . & . & . & . & . & . & 2 & 2 & . & . & 2 & 2 & . & . & 2 & 2 & . & . & 2 & 2 & . & . \\
. & . & . & . & . & . & . & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . & 2 & . \\
-3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

Let $A := \mathrm{Aut}(L) := \{f \in \mathrm{O}(\mathbb{R}^{24}) : f(L) = L\}$, where $\mathrm{O}(\mathbb{R}^n)$ is the orthogonal group (matrices x with $xx^{\mathrm{t}} = 1_n$). According to Exercise 28, A is finite. It holds that $\mathrm{Z}(A) = \langle -1_{24} \rangle$ and one calls $Co_1 := A/\mathrm{Z}(A)$ the first *Conway group* (A is a Schur extension of Co_1). The following vectors lie in L:

$$\begin{aligned} u &:= (4, 4, 0, \ldots, 0), \\ v &:= (4, -4, 0, \ldots, 0), \\ w &:= (5, 1, \ldots, 1). \end{aligned}$$

One calls $Co_2 := \mathrm{C}_A(u)$ and $Co_3 := \mathrm{C}_A(w)$ the second and third Conway group, respectively. Furthermore, $McL := \mathrm{C}_A(u, w)$ is the *McLaughlin group* and $\mathrm{C}_A(v, w) \cong HS$. All are sporadic simple groups with orders

$$\begin{aligned} |Co_1| &= 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 4,157,776,806,543,360,000, \\ |Co_2| &= 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 = 42,305,421,312,000, \\ |Co_3| &= 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 = 495,766,656,000, \\ |McL| &= 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 = 898,128,000. \end{aligned}$$

(vii) Only over 100 years after Mathieu's work did Janko discover the next sporadic simple group J_1. It can be defined most simply by an $\mathbb{F}_{11}$-representation:

$$J_1 := \left\langle \begin{pmatrix}
. & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & . \\
. & . & . & 1 & . & . & . \\
. & . & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . \\
. & . & . & . & . & . & 1 \\
1 & . & . & . & . & . & .
\end{pmatrix}, \begin{pmatrix}
-3 & 2 & -1 & -1 & -3 & -1 & -3 \\
-2 & 1 & 1 & 3 & 1 & 3 & 3 \\
-1 & -1 & -3 & -1 & -3 & -3 & 2 \\
-1 & -3 & -1 & -3 & -3 & 2 & -1 \\
-3 & -1 & -3 & -3 & 2 & -1 & -1 \\
1 & 3 & 3 & -2 & 1 & 1 & 3 \\
3 & 3 & -2 & 1 & 1 & 3 & 1
\end{pmatrix} \right\rangle \leq \mathrm{GL}(7, 11).$$

It holds that $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = 175,560$.

(viii) The second Janko group J_2 (also called *Hall-Janko group* HJ) of order $604,800$ is constructed similarly to HS as a permutation group of degree $100 = 63+36+1$. Let C be the set of involutions and $\mathcal{U}$ the set of subgroups of order 168 of $G = \mathrm{SU}(3,3)$ (both are conjugacy classes). It holds that $|C| = 63$ and $|\mathcal{U}| = |G : U| = 36$ as well as $U \cong \mathrm{GL}(3,2)$ for $U \in \mathcal{U}$. Let Γ be a graph with vertex set $\Gamma_E := C \cup \mathcal{U} \cup \{\gamma\}$, where γ is adjacent to every $U \in \mathcal{U}$. Two involutions $x, y \in C$ are adjacent if and only if $|\langle xy \rangle| = 4$ (i.e., $\langle x, y \rangle \cong D_8$). Two subgroups $U_1, U_2 \in \mathcal{U}$ are adjacent if and only if $|U_1 \cap U_2| = 24$. Finally, $(x, U) \in C \times \mathcal{U}$ form an edge if and only if $x \in U$ holds. Now $\mathrm{Aut}(\Gamma) \cong J_2 \rtimes C_2$ (see Exercise 27). We have thus "constructed" all simple groups of order $\leq 10^7$ (see GT appendix). The sporadic groups J_3 and J_4 are also named after Janko, where J_4 is the last constructed simple group ever (construction 1977 by Norton).

(ix) Presentations, permutation and matrix representations of all sporadic groups can be looked up at `https://brauer.maths.qmul.ac.uk/Atlas/v3/` or obtained directly with GAP:

```
LoadPackage("atlasrep");
DisplayAtlasInfo("M24"); #overview of representations, needs internet
AtlasGroup("M24",Dimension,11,Ring,GF(2));
prog:=AtlasProgram("M24","presentation");;
slp:=StraightLineProgramFromStraightLineDecision(prog.program);;
F:=FreeGroup(2);;
rels:=ResultOfStraightLineProgram(slp,GeneratorsOfGroup(F));;
G:=F/rels;;
PresentationFpGroup(G);

ct:=CharacterTable("M"); #character table of the Monster
PrintFactorsInt(Size(ct)); #prime factorization of the order
NrConjugacyClasses(ct);
SizesConjugacyClasses(ct);
OrdersClassRepresentatives(ct); #orders of elements up to conjugacy
```

(x) Only in 2014 could Keevash show that there exist non-trivial (t, k, v)-Steiner systems with $t \geq 6$. In fact, the divisibility conditions from Lemma 9.18 are in "most" cases sufficient for the existence of a corresponding Steiner system. The proof is probabilistic and non-constructive.[21] The number of $(2, 3, v)$-Steiner systems for $v \leq 19$ can be looked up at OEIS.

10 Coxeter groups

Remark 10.1. The groups that can be generated by two involutions are known to be exactly the dihedral groups. In this section, we investigate groups that can be generated by finitely many involutions. These appear as symmetry groups of higher-dimensional spaces and lead to the classification of (finite) reflection groups.

Definition 10.2. A group of the form

$$G = \langle x_1, \ldots, x_n \mid (x_i x_j)^{m_{ij}} = 1, 1 \leq i \leq j \leq n \rangle$$

with $2 \leq m_{ij} \leq \infty$ for $i < j$ and $m_{ii} = 1$ for $i = 1, \ldots, n$ is called a *Coxeter group* of *rank* n.

[21] For this, a popular science article: wired.com

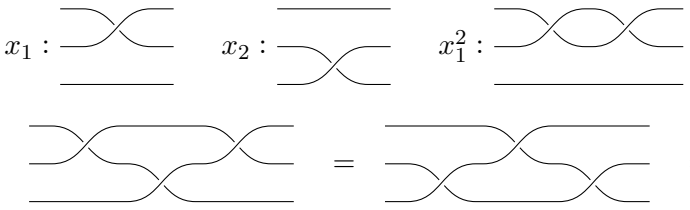
Remark 10.3.

(i) In the following, let $G = \langle x_1, \dots, x_n \rangle$ always be a Coxeter group (we identify the generators $x_1, \dots, x_n$ of the free group with corresponding cosets in G). The relations of the form $(x_i x_j)^\infty = 1$ have no meaning and can be ignored. From $m_{ii} = 1$ it follows that $x_i^2 = 1$ for $i = 1, \dots, n$. Because of $(x_j x_i)^{m_{ij}} = (x_i x_j)^{-m_{ij}} = 1$, we define $m_{ji} := m_{ij}$ for $i < j$. According to von-Dyck, there exists a homomorphism $f\colon G \to \{\pm 1\}$ with $f(x_i) = -1$ for $i = 1, \dots, n$. Thus $x_1, \dots, x_n$ are involutions in G. The equation $m_{ij} = 2$ states that x_i and x_j commute.

(ii) The relation $(x_i x_j)^{m_{ij}} = 1$ is equivalent to $x_i x_j x_i \dots = x_j x_i x_j \dots$, where there are exactly m_{ij} factors on both sides. Let $(x_i x_j)_n := x_i x_j x_i \dots$ be an alternating product of n factors. If one omits the condition $m_{ii} = 2$ in the definition of Coxeter groups, one obtains *Artin groups*:

$$\langle x_1, \dots, x_n \mid (x_i x_j)_{m_{ij}} = (x_j x_i)_{m_{ij}}, 1 \le i, j \le n \rangle$$

with $2 \le m_{ij} \le \infty$. The choice $m_{ij} = \infty$ for all i, j yields the free group F_n. According to von-Dyck, every Coxeter group is a factor group of an Artin group. In the special case $m_{ij} = 3$ for $|i - j| = 1$ and $m_{ij} = 2$ for $|i - j| > 1$, one speaks of a *braid group*. In contrast to S_n (Theorem 2.18), $m_{ii} = \infty$ holds instead of $m_{ii} = 2$. One can realize the elements of the braid group by "braids" with n "strands". The operation is the "gluing" of the strands at their ends:



However, we will not go into these groups any further.

Example 10.4.

(i) $G = \langle x, y \mid x^2 = y^2 = (xy)^m = 1 \rangle \cong D_{2m}$ is a Coxeter group.

(ii) $G = \langle x_1, \dots, x_n \mid x_i^2 = (x_i x_j)^2 = 1, i < j \rangle \cong C_2^n$ is a Coxeter group.

(iii) According to Theorem 2.18, the symmetric groups are Coxeter groups with $x_i = (i, i+1)$ for $i = 1, \dots, n-1$.

(iv) In the case $m_{ij} = \infty$ for all $i < j$, G is called the *universal* Coxeter group of rank n. For $n = 2$, one obtains D_∞. Every Coxeter group is a factor group of a universal Coxeter group.

(v) A *reflection* $\sigma \in \mathrm{GL}(\mathbb{R}^d)$ is a map of the form $\sigma_b(v) := v - 2[v, b]b$ for a normalized vector $b \in \mathbb{R}^d$ (here $[v, b]$ is the standard inner product). A *reflection group* is a finite subgroup $S \le \mathrm{GL}(\mathbb{R}^d)$ generated by reflections $\sigma_1, \dots, \sigma_n$. According to von-Dyck, S is a factor group of a Coxeter group. We will show in Theorem 10.41 that S is indeed isomorphic to a Coxeter group (i. e. all further relations in S follow from the relations $(\sigma_i \sigma_j)^{m_{ij}} = 1$).

(vi) Let S be a finite, non-abelian simple group. According to Feit-Thompson, S possesses an involution s. Obviously, S is generated by all conjugates of s. Thus S is isomorphic to a factor group of a (possibly infinite) Coxeter group.

Definition 10.5. Every $g \in G$ can be written in the form $g = x_{i_1} \dots x_{i_k}$. If k is as small as possible, then this representation is called *reduced* (in contrast to free groups, reduced representations are not necessarily unique). Furthermore, let $l(g) := k$ be the *length* of g.

Lemma 10.6. *For $g, h \in G$ and $1 \leq i \leq n$, the following hold:*

(i) $l(gh) \leq l(g) + l(h)$.

(ii) $l(g^{-1}) = l(g)$.

(iii) $l(gx_i) = l(g) \pm 1$.

Proof. The first two statements are trivial. They show

$$l(g) - 1 = l(gx_ix_i) - l(x_i) \leq l(gx_i) \leq l(g) + l(x_i) = l(g) + 1.$$

The homomorphism $f\colon G \to \{\pm 1\}$ from Remark 10.3 yields

$$(-1)^{l(gx_i)} = f(gx_i) = -f(g) = -(-1)^{l(g)}$$

and $l(gx_i) \neq l(g)$. □

Remark 10.7. The kernel of the homomorphism $f\colon G \to \{\pm 1\}$ is the set of elements of even length. It is called the *alternating subgroup*. In S_n, $\operatorname{sgn}(g) = (-1)^{l(g)}$ holds for $g \in S_n$.

Definition 10.8. Let V be an $\mathbb{R}$-vector space with basis $b_1, \ldots, b_n$.

- We define a symmetric bilinear form on V by

$$[b_i, b_j]_G := [b_i, b_j] = -\cos\frac{\pi}{m_{ij}},$$

  where $-\cos\frac{\pi}{\infty} = -1$ is set.

- For $i = 1, \ldots, n$ let

$$\sigma_i\colon V \to V, \qquad v \mapsto v - 2[v, b_i]b_i.$$

Theorem 10.9. *There exists exactly one homomorphism $\sigma\colon G \to \mathrm{GL}(V)$ with $\sigma(x_i) = \sigma_i$ for $i = 1, \ldots, n$. In this case, $\sigma_i\sigma_j$ has order m_{ij} for $1 \leq i, j \leq n$. With the notation ${}^g v := \sigma(g)(v)$, it holds that $[v, w]_G = [{}^g v, {}^g w]_G$.*

Proof. For $i = 1, \ldots, n$ it holds that $[b_i, b_i] = 1$. Therefore $V_i := b_i^\perp := \operatorname{Ker}(v \mapsto [v, b_i]) \leq V$ is a hyperplane and $V = V_i \oplus \mathbb{R}b_i$. It holds that $\sigma_i(b_i) = -b_i$ and $\sigma_i(v) = v$ for $v \in V_i$. Thus σ_i is a "reflection" at V_i (in contrast to Euclidean space, $[.,.]_G$ is not necessarily positive definite). In particular, σ_i has order 2. Now let $i < j$ and $W := \langle b_i, b_j \rangle$. The definition of σ_i shows that $\langle \sigma_i, \sigma_j \rangle$ acts on W.

First let $m_{ij} = \infty$. Then it holds that

$$(\sigma_i\sigma_j)^k(b_i) = (\sigma_i\sigma_j)^{k-1}(\sigma_i(b_i + 2b_j)) = (\sigma_i\sigma_j)^{k-1}(3b_i + 2b_j) = \ldots = (2k+1)b_i + 2kb_j$$

for $k \geq 1$. In particular, $\sigma_i\sigma_j$ has infinite order.

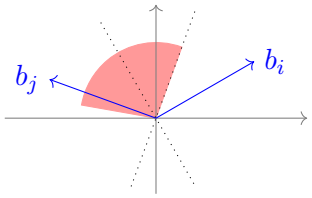
Now let $m_{ij} < \infty$ and $\varphi := \pi/m_{ij}$. For $v = \lambda b_i + \mu b_j \in W$ it holds that

$$[v, v] = \lambda^2 - 2\lambda\mu\cos\varphi + \mu^2 = (\lambda - \mu\cos\varphi)^2 + \mu^2\sin(\varphi)^2 > 0.$$

Thus $[.,.]_G$ is positive definite on W and therefore coincides with the standard scalar product up to a choice of basis. Furthermore,

$$[b_i, b_j] = -\cos\varphi = \cos(\pi - \varphi),$$

d. h. the "angle" between b_i and b_j is $\pi - \varphi$. Therefore $(\sigma_i\sigma_j)_{|W}$ is a "rotation" by 2φ (since the map is linear, it suffices to consider the images of the two linearly independent reflection axes):



In particular, $(\sigma_i\sigma_j)_{|W}$ has order m_{ij}. Since $[.,.]_G$ is positive definite on W, it holds that $V = W \oplus W^\perp = W \oplus (V_i \cap V_j)$. Since $\sigma_i\sigma_j$ acts trivially on $V_i \cap V_j$, $\sigma_i\sigma_j$ also has order m_{ij} on V. By von-Dyck, σ is now a homomorphism. For the last statement we calculate

$$\begin{aligned}[\sigma_i(v), \sigma_i(w)] &= [v - 2[v, b_i]b_i, w - 2[w, b_i]b_i] \\ &= [v, w] - 2[v, b_i][b_i, w] - 2[w, b_i][v, b_i] + 4[v, b_i][w, b_i] = [v, w].\end{aligned}$$ □

Definition 10.10.

- One calls
$$\Phi := \{{}^g b_i : 1 \leq i \leq n, g \in G\} \subseteq V$$
the *root system* of G and its elements are called *roots*.
- A root v can be uniquely written in the form $v = \sum_{i=1}^n v_i b_i$. One calls v *positive* (resp. *negative*), if $v_1, \ldots, v_n \geq 0$ (resp. $v_1, \ldots, v_n \leq 0$). If applicable, one writes $v > 0$ (resp. $v < 0$; the case $v = 0$ is excluded). Let the set of positive roots be Π.
- For $v \in \Phi$ let
$$\sigma_v \colon V \to V, \qquad w \mapsto w - 2[w, v]v.$$

Remark 10.11.

(i) According to Theorem 10.9, all roots of G are normalized with respect to $[.,.]_G$. In particular, $\sigma_v(v) = -v$, d. h. σ_v is the reflection at the hyperplane $v^\perp$.

(ii) Because of $\sigma_i(b_i) = -b_i$, we have $-\Phi = \Phi$ and $\sigma_v = \sigma_{-v}$. Let $g \in G$ and $1 \leq i \leq n$ with $v = g(b_i)$. Then
$${}^{gx_ig^{-1}}w = {}^g({}^{g^{-1}}w - 2[{}^{g^{-1}}w, b_i]b_i) = w - 2[w, {}^g b_i]{}^g b_i = w - 2[w, v]v.$$
This shows $\sigma(gx_ig^{-1}) = \sigma_v$. We will also refer to $x_v := gx_ig^{-1}$ as a reflection.

Example 10.12.

(i) Let $G = \langle x, y\rangle \cong D_{2m}$, $\varphi = \pi - \frac{\pi}{m}$, $b_1 = (1, 0)$ and $b_2 = (\cos\varphi, \sin\varphi)$. Then $[.,.]_G$ is the standard inner product on $\mathbb{R}^2$ with respect to b_1, b_2 and $\sigma \colon G \to \mathrm{GL}(\mathbb{R}^2)$ is the well-known representation as the symmetry group of the regular m-gon. The roots correspond to the $2m$ reflection axes. Geometrically, the positive roots lie "between" b_1 and b_2. In particular, $\Phi = \Pi \cup (-\Pi)$.

(ii) Let $G = \langle x_1, \ldots, x_{n-1} \rangle \cong S_n$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and $b_i := \frac{1}{\sqrt{2}}(e_i - e_{i+1})$ for $i = 1, \ldots, n-1$. It is easy to verify that $[.,.]_G$ is the standard inner product on $V := \langle b_1, \ldots, b_{n-1} \rangle$ with respect to $b_1, \ldots, b_{n-1}$. The homomorphism $\sigma\colon G \to \mathrm{GL}(V)$ arises from the permutation action of S_n on the n coordinates, because

$$\begin{aligned}
{}^{x_i}b_i = \sigma_i(b_i) = -b_i = \frac{1}{\sqrt{2}}(e_{i+1} - e_i) &= \frac{1}{\sqrt{2}}(e_{x_i(i)} - e_{x_i(i+1)}),\\
{}^{x_i}b_{i+1} = b_{i+1} - 2[b_{i+1}, b_i]b_i = b_{i+1} + b_i = \frac{1}{\sqrt{2}}(e_i - e_{i+2}) &= \frac{1}{\sqrt{2}}(e_{x_i(i+1)} - e_{x_i(i+2)}),\\
{}^{x_i}b_j = b_j = \frac{1}{\sqrt{2}}(e_{x_i(j)} - e_{x_i(j+1)}) &\qquad (|j-i| > 1).
\end{aligned}$$

Obviously, $\Phi = \{e_i - e_j : i \neq j\}$ and $\Pi = \{e_i - e_j : i < j\}$, because $e_i - e_j = b_i + b_{i+1} + \ldots + b_{j-1}$. In particular, $|\Phi| = n(n-1)$ and $|\Pi| = n(n-1)/2$.

Lemma 10.13. *For $g \in G$ and $1 \leq i \leq n$, it holds that ${}^g b_i > 0$ if $l(gx_i) > l(g)$ and ${}^g b_i < 0$ if $l(gx_i) < l(g)$. In particular, every root is positive or negative, d. h. $\Phi = \Pi \cup (-\Pi)$.*

Proof. Due to $g = gx_ix_i$ and ${}^{gx_i}b_i = -{}^g b_i$, it suffices to consider the case $l(gx_i) > l(g)$. We prove the first statement by induction on $l(g)$. In the case $l(g) = 0$, we have $g = 1$, $l(x_i) = 1$ and $b_i > 0$. Now let $g \neq 1$ and $1 \leq j \leq n$ with $l(gx_j) = l(g) - 1$ (the last factor of a reduced representation of g). By assumption, $x_j \neq x_i$. We consider the dihedral group $H := \langle x_i, x_j \rangle \leq G$. Let $l_H\colon H \to \mathbb{N}_0$ be the length function with respect to the generators x_i, x_j of H.[22] Let

$$A := \{y \in gH : l(y) + l_H(y^{-1}g) = l(g)\}.$$

Since $g \in A$, we have $A \neq \varnothing$. Choose $y \in A$ with $l(y)$ minimal. Because $l(gx_j) + l_H(x_j) = l(g) - 1 + 1 = l(g)$, it follows that $gx_j \in A$. The choice of y shows $l(y) \leq l(gx_j) < l(g)$.

Suppose $l(yx_i) = l(y) - 1$. Then

$$l(g) \leq l(yx_i) + l_H(x_i y^{-1} g) \leq l(y) - 1 + l_H(y^{-1}g) + 1 = l(y) + l_H(y^{-1}g) = l(g)$$

and $l(g) = l(yx_i) + l_H(x_i y^{-1} g)$. This shows $yx_i \in A$ in contradiction to $l(yx_i) < l(y)$. Thus $l(yx_i) > l(y)$ and completely analogously $l(yx_j) > l(y)$. Induction shows ${}^y b_i, {}^y b_j > 0$. Let $h := y^{-1}g \in H$. Since $g = yh$, it suffices to show that ${}^h b_i$ is a non-negative linear combination of b_i and b_j.

Suppose $l_H(hx_i) < l_H(h)$. Then

$$l(gx_i) = l(yhx_i) \leq l(y) + l_H(hx_i) < l(y) + l_H(h) = l(g).$$

Thus every reduced representation of h with respect to x_i, x_j must end in x_j. In the case $m_{ij} = \infty$, we have ${}^{x_j}b_i = b_i + 2b_j$, ${}^{x_ix_j}b_i = {}^{x_i}(b_i + 2b_j) = 3b_i + 2b_j$ etc. (cf. proof of Theorem 10.9). We can therefore assume $m := m_{ij} < \infty$. Then b_i and b_j form the angle $\pi - \frac{\pi}{m}$ and x_ix_j is a "rotation" by $\frac{2\pi}{m}$. If $l_H(h) = m$, then $h = (x_ix_j)^{m/2}$ if m is even and $h = x_j(x_ix_j)^{(m-1)/2}$ if m is odd. In both cases there would be a reduced representation ending with x_i (namely $h = (x_jx_i)^{m/2}$ or $x_i(x_jx_i)^{(m-1)/2}$). Thus $l_H(h) < m$ and $h \in \{(x_ix_j)^k, x_j(x_ix_j)^k\}$ with $k < m/2$. In the case $k = (m-1)/2$, we have $h = (x_ix_j)^k$ and ${}^h b_i = b_j$. In all other cases, $(x_ix_j)^k$ is a rotation by less than $\frac{(m-1)\pi}{m}$, i. e. ${}^{(x_ix_j)^k}b_i$ lies strictly "between" b_i and b_j. A further application of x_j does not change this. Thus ${}^h b_i$ is a non-negative linear combination of b_i and b_j. □

[22] According to Lemma 10.17, l_H is the restriction of l to H. However, this is not used here.

Theorem 10.14. *The homomorphism $\sigma\colon G \to \mathrm{GL}(V)$ from Theorem 10.9 is injective.*

Proof. Let $g \in \mathrm{Ker}(\sigma) \setminus \{1\}$. Then there exists $1 \leq i \leq n$ with $l(gx_i) < l(g)$. Lemma 10.13 yields the contradiction $b_i = {}^g b_i < 0$. □

Definition 10.15. For $I \subseteq \{1, \ldots, n\}$, $G_I := \langle x_i : i \in I \rangle \leq G$ is called a *parabolic* subgroup of G.

Theorem 10.16. *For $I \subseteq \{1, \ldots, n\}$,*

$$G_I \cong \langle \{y_i : i \in I\} \mid \{(y_i y_j)^{m_{ij}} : i, j \in I\} \rangle,$$

d. h. G_I is itself a Coxeter group.

Proof. Applying the construction of σ to the Coxeter group on the right-hand side, one obtains exactly $\sigma(G_I) \cong G_I$. □

Lemma 10.17. *Let $g = x_{i_1} \ldots x_{i_k} \in G_I$ be reduced in G. Then $i_1, \ldots, i_k \in I$ holds. In particular, $\{x_1, \ldots, x_n\} \cap G_I = \{x_i : i \in I\}$.*

Proof. Induction on k. Wlog. let $k \geq 1$. According to Lemma 10.13, ${}^g b_{i_k} < 0$ holds. Furthermore, let $g = x_{j_1} \ldots x_{j_l}$ with $j_1, \ldots, j_l \in I$. By the definition of σ_i, it holds that

$${}^g b_{i_k} = b_{i_k} + \sum_{a=1}^{l} \lambda_a b_{j_a}$$

with $\lambda_a \in \mathbb{R}$. Because ${}^g b_{i_k} < 0$, $i_k = j_s \in I$ must hold for some $1 \leq s \leq l$. In particular, $h := x_{i_1} \ldots x_{i_{k-1}} = g x_{i_k} \in G_I$ is reduced. The claim now follows by induction. □

Corollay 10.18. *If $g = x_{i_1} \ldots x_{i_k} = x_{j_1} \ldots x_{j_k}$ are two reduced representations of $g \in G$, then $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}$ holds.*

Example 10.19. In the situation of Corollay 10.18, $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$ do not have to coincide as multisets. In S_3, for example, $x_1 x_2 x_1 = (1,3) = x_2 x_1 x_2$ are reduced representations.

Theorem 10.20. *We consider S_n as a Coxeter group in the generators $x_i := (i, i+1)$. For $\sigma \in S_n$, there exist uniquely determined numbers $l \geq 0$ and $1 \leq a_1, \ldots, a_l \leq n-1$ with the following properties:*

(i) $\sigma = x_{a_1} \ldots x_{a_l}$.

(ii) $a_i \neq a_{i-1} \leq a_i + 1$ *for* $i = 2, \ldots, l$.

(iii) The sequence $(a_1, \ldots, a_l)$ has no segment of the form $(a, a-1, \ldots, a-r, a)$ with $r \geq 1$.

If applicable, $\sigma = x_{a_1} \ldots x_{a_l}$ is a reduced word, i. e., $l = l(\sigma)$.

Proof. We apply the following algorithm to an arbitrary representation $\sigma = x_{a_1} \ldots x_{a_k}$:

(1) If $a_i = a_{i+1}$ for some i, then delete $x_{a_i} x_{a_{i+1}} = 1$ from the representation.

(2) If $a_{i-1} > a_i + 1$, then swap $x_{a_{i-1}}$ and x_{a_i} and start again at (1). This is allowed since $x_{a_{i-1}}$ and x_{a_i} are disjoint.

(3) If $(a_1, \ldots, a_k)$ contains a segment of the form $(a, a-1, \ldots, a-r, a)$, then replace it with $(a-1, a, a-1, \ldots, a-r)$ and start again at (1). This does not change σ, because

$$x_a x_{a-1} \ldots x_{a-r} x_a = x_a x_{a-1} x_a x_{a-2} x_{a-3} \ldots x_{a-r} = x_{a-1} x_a x_{a-1} x_{a-2} \ldots x_{a-r}$$

according to the braid relation.

Through (1) and (3), $\sum_{i=1}^{k} a_i$ is reduced, while (2) decreases the lexicographical order of the sequence $(a_1, \ldots, a_k)$. Therefore, the algorithm must terminate after finitely many steps. At the end, conditions (i)–(iii) are satisfied.

A sequence $(a_1, \ldots, a_l)$ that satisfies (ii) and (iii) is called *regular*. For $n = 2$, there are only the regular sequences $a = ()$ (with $l = 0$) and $a = (1)$. Inductively, we assume that there are exactly $(n-1)!$ regular sequences $(a_1, \ldots, a_l)$ with $1 \leq a_1, \ldots, a_l \leq n-2$. Now let $(a_1, \ldots, a_l)$ be such that $a_k = n-1$. Then $(a_k, a_{k+1}, \ldots, a_l) = (n-1, n-2, \ldots, n-r)$ for some $r \geq 1$ due to (ii) and (iii). For $(a_1, \ldots, a_{k-1})$, there are inductively exactly $(n-1)!$ possibilities, while for r there are exactly $n-1$ possibilities. Therefore, there exist $(n-1)!(n-1)$ regular sequences that contain $n-1$. Together with the $(n-1)!$ sequences that do not contain $n-1$, one obtains exactly $n! = |S_n|$ regular sequences. Thus, each permutation can be represented by only one regular sequence.

If one applies the above algorithm to a reduced word, the length l cannot become smaller. Thus, $l = l(\sigma)$ holds. □

Example 10.21. In S_5, we have

$$\begin{aligned} x_4 x_3 x_2 x_3 x_1 x_4 x_2 &= x_4 x_3 (x_2 x_1 x_2) x_3 x_4 = x_4 x_3 (x_1 x_2 x_1) x_3 x_4 = x_1 x_4 (x_3 x_2 x_1 x_3) x_4 \\ &= x_1 x_4 (x_2 x_3 x_2 x_1) x_4 = x_1 x_2 (x_4 x_3 x_2 x_1 x_4) = x_1 x_2 x_3 x_4 x_3 x_2 x_1. \end{aligned}$$

A reduced representation can be found somewhat faster by proceeding recursively. Let $\sigma \in S_n$ and $a := \sigma^{-1}(n)$. Then $\tau := \sigma x_a x_{a+1} \ldots x_{n-1} \in S_{n-1}$. By induction, τ has a representation in the desired form. Therefore, $\sigma = \tau t_{n-1} t_{n-2} \ldots t_a$ also has this property.

Theorem 10.22. *The map $I \mapsto G_I$ is an isomorphism of lattices, d. h. it holds that*

(i) $I \subseteq J \iff G_I \leq G_J$.

(ii) $G_{I \cup J} = \langle G_I, G_J \rangle$.

(iii) $G_{I \cap J} = G_I \cap G_J$.

Proof.

(i) From $I \subseteq J$ it follows obviously that $G_I \leq G_J$. If $G_I \leq G_J$, then it follows

$$\{x_i : i \in I\} = \{x_1, \ldots, x_n\} \cap G_I \subseteq \{x_1, \ldots, x_n\} \cap G_J = \{x_j : j \in J\}$$

from Lemma 10.17.

(ii) Trivial.

(iii) Obviously $G_{I \cap J} \leq G_I \cap G_J$. The reverse inclusion follows from Lemma 10.17. □

Theorem 10.23. *For $g \in G$, $l(g)$ is the number of positive roots that are mapped to negative roots under g, d. h. $l(g) = |\Pi \cap g^{-1}(-\Pi)| \leq |\Pi|$. In particular, this number is finite.*

Proof. Induction on $l(g)$: Wlog. $g \neq 1$. Let first $g = x_i$ for some $1 \leq i \leq n$. Because of ${}^g b_i = -b_i$ we must show ${}^g v > 0$ for all $v \in \Pi \setminus \{b_i\}$. Since all roots are normalized, v is not a multiple of b_i. So let $v = \sum_{j=1}^n \lambda_j b_j$ with $\lambda_k > 0$ for at least one $k \neq i$. In ${}^g v = \sigma_i(v) = v - 2[v, b_i]b_i$, b_k still has the positive coefficient λ_k. Therefore ${}^g v > 0$.

Now let $l(g) \geq 2$ and $1 \leq i \leq n$ with $l(gx_i) = l(g) - 1$. According to Lemma 10.13 we have ${}^g b_i < 0$, d. h. $b_i \in \Pi \cap g^{-1}(-\Pi)$. With what was just proven, it follows

$$\Pi \cap (gx_i)^{-1}(-\Pi) = x_i(x_i(\Pi) \cap g^{-1}(-\Pi)) = x_i(\Pi \cap g^{-1}(-\Pi) \setminus \{b_i\}).$$

Induction shows

$$|\Pi \cap g^{-1}(-\Pi)| = |\Pi \cap (gx_i)^{-1}(-\Pi)| + 1 = l(gx_i) + 1 = l(g). \qquad \square$$

Remark 10.24. If the length in G is bounded, say by m, then $|G| \leq n^m < \infty$. In infinite Coxeter groups there must therefore be infinitely many (positive) roots. In particular, $-\operatorname{id}_V \notin \sigma(G)$ (otherwise $-\operatorname{id}$ would have infinite length).

Corollay 10.25. *If G is finite, then there exists exactly one element $g \in G$ with maximal length.*

Proof. Suppose $g, h \in G$ have maximal length $l(g) = l(h)$. Then $l(gx_i) < l(g)$ and ${}^g b_i < 0$ for $i = 1, \ldots, n$. Since every positive root is a non-negative linear combination of the b_i, g (and h) must map all positive roots to negative roots, d. h. $g(\Pi) = -\Pi$. Thus $g^2(\Pi) = \Pi = gh(\Pi)$ and $l(g^2) = 0 = l(gh)$. This shows $h = g^{-1} = g$. $\square$

Example 10.26.

(i) For $G = \langle x, y \rangle \cong D_{2m}$, $z := xy \ldots = yx \ldots$ (m factors each) is the element of maximal length (cf. Example 10.12). If m is even, then $\sigma(z) = -\operatorname{id}$, i.e., the rotation by π.

(ii) For $g \in G = S_n$ and $i < j$ we have

$${}^g\big(\frac{1}{\sqrt{2}}(e_i - e_j)\big) < 0 \iff \frac{1}{\sqrt{2}}(e_{g(i)} - e_{g(j)}) < 0 \iff g(i) > g(j)$$

(Example 10.12). Therefore $l(g)$ is the number of *inversions* of g, i.e., pairs $i < j$ with ${}^g i > {}^g j$. The element of maximal length is therefore

$$g = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} = (1, n)(2, n-1) \ldots = t_1 \ldots t_{n-1}$$

with $t_i = (i+1, i) \ldots (2, 1)$ and $l(g) = 1 + \ldots + n - 1 = n(n-1)/2 = |\Pi|$.

Lemma 10.27 (Tits)**.** *Let $g \in G$ and $v \in \Pi$. Then $l(gx_v) > l(g) \iff {}^g v > 0$.*

Proof. As in Lemma 10.13, it suffices to show $l(gx_v) > l(g) \Rightarrow {}^g v > 0$. Induction on $l(g)$. The case $g = 1$ is clear. Now let $l(g) > 0$ and $l(x_i g) < l(g)$. Because of

$$l(x_i g x_v) \geq l(g x_v) - 1 > l(g) - 1 = l(x_i g)$$

it holds that ${}^{x_i g} v > 0$ by the induction hypothesis. Let us assume ${}^g v < 0$. According to Theorem 10.23, it then follows that ${}^g v = -b_i$. This shows ${}^{g^{-1} x_i g} v = {}^{g^{-1}} b_i = -v$ and $g^{-1} x_i g = x_v$ according to Remark 10.11. However, this contradicts $l(gx_v) > l(g) > l(x_i g) = l(gx_v)$. $\square$

Remark 10.28. In the following, we use the notation $x_1 \dots \check{x}_i \dots x_k := x_1 \dots x_{i-1} x_{i+1} \dots x_k$.

Theorem 10.29.

(i) (Exchange condition) *Let $g = x_{i_1} \dots x_{i_k} \in G$ and $v \in \Phi$ with $l(gx_v) < l(g)$. Then there exists $1 \leq s \leq k$ with $gx_v = x_{i_1} \dots \check{x}_{i_s} \dots x_{i_k}$. If $l(g) = k$, then s is uniquely determined.*

(ii) (Deletion condition) *Let $g = x_{i_1} \dots x_{i_k} \in G$ with $l(g) < k$. Then there exist $1 \leq s < t \leq k$ with $g = x_{i_1} \dots \check{x}_{i_s} \dots \check{x}_{i_t} \dots x_{i_k}$.*

Proof.

(i) Because of $x_v = x_{-v}$, we can assume $v > 0$. From Lemma 10.27 it follows that ${}^g v < 0$. Because of $v > 0$, there exists an s with ${}^{x_{i_{s+1}} \dots x_{i_k}} v > 0$ and ${}^{x_{i_s} \dots x_{i_k}} v < 0$. From Theorem 10.23 it follows that ${}^{x_{i_{s+1}} \dots x_{i_k}} v = b_{i_s}$. This shows $(x_{i_{s+1}} \dots x_{i_k}) x_v (x_{i_{s+1}} \dots x_{i_k})^{-1} = x_{i_s}$, hence $gx_v = x_{i_1} \dots \check{x}_{i_s} \dots x_{i_k}$. Now let $l(g) = k$. Suppose there exist $s < t$ with $x_{i_1} \dots \check{x}_{i_s} \dots x_{i_k} = gx_v = x_{i_1} \dots \check{x}_{i_t} \dots x_{i_k}$. This yields $x_{i_{s+1}} \dots x_{i_t} = x_{i_s} \dots x_{i_{t-1}}$ and $x_{i_s} \dots x_{i_t} = x_{i_{s+1}} \dots x_{i_{t-1}}$. But then $g = x_{i_1} \dots \check{x}_{i_s} \dots \check{x}_{i_t} \dots x_{i_k}$ and $l(g) < k$.

(ii) Because of $l(g) < k$, there exists a t with $l(x_{i_1} \dots x_{i_t}) < l(x_{i_1} \dots x_{i_{t-1}})$. From (i) it follows that $x_{i_1} \dots x_{i_t} = x_{i_1} \dots \check{x}_{i_s} \dots x_{i_{t-1}}$ for some $s < t$. □

Remark 10.30.

(i) It holds that

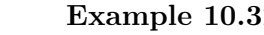
$$l(x_v g) < l(g) \Longrightarrow l(g^{-1} x_v) < l(g^{-1}) \Longrightarrow g^{-1} x_v = x_{i_k} \dots \check{x}_{i_s} \dots x_{i_1} \Longrightarrow gx_v = x_{i_1} \dots \check{x}_{i_s} \dots x_{i_k}.$$

(ii) One can show that every group that satisfies the exchange condition (or deletion condition) with respect to a generating set of involutions is a Coxeter group.

(iii) The deletion condition shows that from an arbitrary representation $g = x_{i_1} \dots x_{i_k}$, one obtains a reduced representation by suggestive deletion.

Definition 10.31.

- G is called *irreducible*, if no partition $\{1, \dots, n\} = I \dot{\cup} J$ with $G = G_I \times G_J$ exists.
- The *Coxeter graph* $C(G)$ consists of the vertices $e_1, \dots, e_n$ and the edges (e_i, e_j) with $m_{ij} \geq 3$. In the case $m_{ij} > 3$, the edges are labeled with m_{ij}. Obviously, G is uniquely determined by $C(G)$ up to the order of the x_i.

Example 10.32. It holds that $C(S_n)$:  and $C(D_{2m})$: according to Example 10.4.

Theorem 10.33. *G is irreducible if and only if $C(G)$ is connected.*

Proof. If $G = G_I \times G_J$, then $m_{ij} = 2$ for $i \in I$ and $j \in J$. Thus there is no path between e_i and e_j in $C(G)$. Conversely, if $C(G)$ is disconnected, then there exists a partition $\{1, \dots, n\} = I \cup J$ with $[x_i, x_j] = 1$ (commutator) for all $i \in I$ and $j \in J$. In particular, $[G_I, G_J] = 1$. According to Theorem 10.22, $G = G_{I \cup J} = \langle G_I, G_J \rangle$ and $G_I \cap G_J = G_{I \cap J} = 1$. This shows $G = G_I \times G_J$. □

Remark 10.34. Attention: The isomorphism type of G does not determine whether G is irreducible as a Coxeter group. For example, $G = D_{12}$ is irreducible with two generators, but also reducible with three generators $G \cong D_6 \times S_2$.

Lemma 10.35. *Let G be irreducible and $V_0 := \{v \in V : [v, V]_G = 0\} \leq V$. Then G acts trivially on V_0 and every proper G-invariant subspace of V lies in V_0.*

Proof. For $v \in V_0$, it holds that ${}^{x_i}v = \sigma_i(v) = v - 2[v, b_i]_G b_i = v$. Since G is generated by $x_1, \ldots, x_n$, G acts trivially on V_0.

Now let $W < V$ be G-invariant. Suppose there exists $v \in W \setminus V_0$. Then there exists an i with $[v, b_i]_G \neq 0$. It follows that $b_i = \frac{v - \sigma_i(v)}{2[v,b_i]} \in W$. For b_j with $m_{ij} \geq 3$, it holds that $[b_i, b_j]_G \neq 0$ and $b_j = \frac{b_i - \sigma_j(b_i)}{2[b_i,b_j]} \in W$. Since $C(G)$ is connected, one obtains $b_1, \ldots, b_n \in W$ in contradiction to $W < V$. □

Theorem 10.36. *Let G be a finite irreducible Coxeter group and $z \in G$ with maximal length. Then $\mathrm{Z}(G) \leq \langle z \rangle$ holds. In particular, $|\mathrm{Z}(G)| \leq 2$.*

Proof. Let $g \in \mathrm{Z}(G) \setminus \{1\}$. From ${}^{x_i}({}^g b_i) = {}^{gx_i} b_i = -{}^g b_i$ it follows that ${}^g b_i = \pm b_i$ (Theorem 10.23). Therefore $b_i \in E_1(\sigma(g)) \cup E_{-1}(\sigma(g))$ (eigenspaces for the eigenvalue 1 and -1, respectively). Because $g \in \mathrm{Z}(G)$, $E_1(\sigma(g))$ and $E_{-1}(\sigma(g))$ are G-invariant and $E_1(\sigma(g)) < V$, since $g \neq 1$. In the case $E_{-1}(\sigma(g)) < V$, it would follow that $b_i \in V_0$ according to Lemma 10.35. However, $[b_i, b_i]_G = 1$. This shows $\sigma(g) = -\mathrm{id}$ and ${}^g v < v$ for all $v \in \Pi$. From Theorem 10.23 it follows that $g = z$. □

Lemma 10.37. *Let $H \leq \mathrm{GL}(n, \mathbb{R})$ be finite and irreducible as a matrix group. Then:*

(i) There exists an H-invariant positive definite bilinear form on $\mathbb{R}^n$.

(ii) Assume there exist $h \in H$ and $\lambda \in \mathbb{R}$ such that the eigenspace $E_\lambda(h)$ has odd dimension. Then $\mathrm{C}_{\mathrm{GL}(n,\mathbb{R})}(H) = \mathbb{R}^\times 1_n$.[23] In particular, this holds if n is odd.

(iii) If $\mathrm{C}_{\mathrm{GL}(n,\mathbb{R})}(H) = \mathbb{R}^\times 1_n$, then any two non-degenerate H-invariant bilinear forms on $\mathbb{R}^n$ differ only by a constant.

Proof.

(i) For $v, w \in \mathbb{R}^n$,

$$[v, w]_H := \sum_{h \in H} [hv, hw]$$

defines an H-invariant positive definite bilinear form, where $[.,.]$ is the standard scalar product.

(ii) Let $f \in \mathrm{C}_{\mathrm{GL}(n,\mathbb{R})}(H)$. Then f operates on $E_\lambda(h)$. Since $\dim E_\lambda(h)$ is odd, f possesses a real eigenvalue μ on $E_\lambda(h)$. Now $0 \neq E_\mu(f) \leq \mathbb{R}^n$ is H-invariant and it follows that $E_\mu(f) = \mathbb{R}^n$ as well as $f = \mu 1_n$, since H is irreducible. The second statement is obtained with $h = 1$.

[23] One says: H is *absolutely* irreducible.

(iii) Let $[.,.]_1$ and $[.,.]_2$ be two non-degenerate H-invariant bilinear forms on $V := \mathbb{R}^n$. Let $V^* := \mathrm{Hom}(V, \mathbb{R})$ be the dual space of V. Then $\varphi_i \colon V \to V^*$, $v \mapsto [v,.]_i$ for $i = 1, 2$ are isomorphisms of vector spaces (note $\dim V = \dim V^*$). Thus $f := \varphi_2^{-1} \circ \varphi_1 \colon V \to V$ is an isomorphism with $[v, w]_1 = [f(v), w]_2$ for all $v, w \in V$. For $h \in H$ it follows that

$$[f(hv), hw]_2 = [hv, hw]_1 = [v, w]_1 = [f(v), w]_2 = [hf(v), hw]_2.$$

This shows $fh = hf$, i. e. $f \in \mathrm{C}_{\mathrm{GL}(n,\mathbb{R})}(H) = \mathbb{R}^\times 1_n$. □

Remark 10.38. We consider the dual space $V^* := \mathrm{Hom}(V, \mathbb{R})$ with the dual basis $\beta_1, \ldots, \beta_n$, where $\beta_i(b_j) = \delta_{ij}$. Through ${}^g\varphi(v) := \varphi({}^{g^{-1}}v)$ for $v \in V$, $\varphi \in V^*$ and $g \in G$, G operates on V^*. The map $\Gamma \colon V \to V^*$, $v \mapsto [v,.]_G$ provides an isomorphism between the actions on V and V^*, because

$$({}^g\Gamma(v))(w) = \Gamma(v)({}^{g^{-1}}w) = [v, {}^{g^{-1}}w]_G = [{}^g v, w]_G = \Gamma({}^g v)(w)$$

for $v, w \in V$ and $g \in G$. Let $\sigma^* \colon G \to \mathrm{GL}(V^*)$ be the corresponding monomorphism. We define

$$C := \{\varphi \in V^* : \forall i : \varphi(b_i) > 0\} \subseteq V^*.$$

With respect to the dual basis, $C = \mathbb{R}^n_{>0}$ is an open set in the Euclidean space $\mathbb{R}^n$. Since the determinant $\mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous, $\mathrm{GL}(V^*) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is open in $\mathbb{R}^{n\times n}$.

Theorem 10.39. *The image $\sigma^*(G)$ is a discrete subgroup of $\mathrm{GL}(V^*)$, i.e., for all $a \in \sigma^*(G)$ there exists an open neighborhood $U(a) \subseteq \mathrm{GL}(V^*)$ with $U(a) \cap \sigma^*(G) = \{a\}$. In particular, $\sigma^*(G)$ is closed.*

Proof. We identify G with $\sigma^*(G)$. Let $c \in C$ and $F \colon \mathrm{GL}(V^*) \to V^*$, $a \mapsto {}^a c = a(c)$. Since matrix-vector multiplication is continuous, F is continuous and $D := F^{-1}(C) \subseteq \mathrm{GL}(V^*)$ is an open neighborhood of $1 \in \mathrm{GL}(V^*)$. For $g \in G \setminus \{1\}$ there exists x_i with $l(g^{-1}x_i) < l(g)$. From Tits Lemma it follows that ${}^{g^{-1}}b_i < 0$ and ${}^g c(b_i) = c({}^{g^{-1}}b_i) < 0$. This shows $D \cap G = \{1\}$. For an arbitrary $g \in G$, gD is an open neighborhood of g with $gD \cap G = g(D \cap G) = \{g\}$.

Now let $(a_i) \subseteq G$ be a convergent sequence. Then there exists $k \in \mathbb{N}$ with $a_i \in U(a_k) \cap G = \{a_k\}$ for all $i \geq k$. Thus the sequence becomes constant and the limit lies in G. This shows that G is closed. □

Theorem 10.40. *G is finite if and only if $[.,.]_G$ is positive definite on V.*

Proof. Let G be finite. We argue by induction on n. In the case $n = 1$, $[b_1, b_1]_G = 1$ and we are finished. Let $n > 1$. Assume that $G = G_I \times G_J$ is reducible. Then $V = V_I \oplus V_J$ with $V_I := \langle b_i : i \in I \rangle$ and analogously V_J. By induction, $[.,.]_{G_I}$ and $[.,.]_{G_J}$ are positive definite on V_I and V_J, respectively. For $i \in I$ and $j \in J$, we have $[b_i, b_j]_G = -\cos\frac{\pi}{2} = 0$. For $v = v_I + v_J \neq 0$, it follows that

$$[v, v]_G = [v_I, v_I]_{G_I} + [v_J, v_J]_{G_J} > 0.$$

Thus $[.,.]_G$ is positive definite on V.

Now let G be irreducible and $V_0 = \{v \in V : [v, V]_G = 0\} \leq V$. By Maschke, V_0 has a G-invariant complement $W \leq V$. Lemma 10.35 shows $W = V$ and $V_0 = 0$, i. e. $[.,.]_G$ is non-degenerate. The same argument also shows that $\sigma(G)$ is irreducible as a matrix group. Because $\dim E_{-1}(\sigma(x_1)) = \dim E_{-1}(\sigma_1) = 1$, $[.,.]_G$ is uniquely determined up to a constant by Lemma 10.37. However, Lemma 10.37 also states that a G-invariant positive definite bilinear form exists. Thus $[.,.]_G$ is positive definite.

Conversely, let $[.,.]_G$ be positive definite. Through the isomorphism $\Gamma\colon V \to V^*$, $v \mapsto [v,.]_G$ from Remark 10.38, one obtains a G-invariant positive definite bilinear form on V^* (namely $[\varphi,\mu] := [\Gamma^{-1}(\varphi),\Gamma^{-1}(\mu)]_G$ for $\varphi,\mu \in V^*$). By Sylvester's law of inertia, V^* has an orthonormal basis Δ with respect to this bilinear form. Writing $g \in \sigma^*(G)$ as a matrix with respect to Δ, the columns have norm 1. Since all norms on $\mathbb{R}^n$ are equivalent, $\sigma^*(G)$ is bounded (with respect to the Euclidean norm). According to Theorem 10.39, $\sigma^*(G)$ is additionally closed and therefore compact. For $a \in \sigma^*(G)$, we choose according to Theorem 10.39 an open neighborhood $U(a) \subseteq \mathbb{R}^{n\times n}$ with $U(a) \cap \sigma^*(G) = \{a\}$. By Heine-Borel, a finite selection of these neighborhoods already covers $\sigma^*(G)$. Therefore $\sigma^*(G) \cong G$ must be finite. □

Theorem 10.41. *The finite Coxeter groups are exactly the reflection groups.*

Proof. Let G be a finite Coxeter group. According to Theorem 10.9 and Theorem 10.14, $G \cong \sigma(G) \leq \mathrm{GL}(V)$ is a reflection group.

Conversely, let $V := \mathbb{R}^n$ and $S \leq \mathrm{GL}(V)$ be a reflection group. Let $\Phi \subseteq V$ be the set of unit vectors b such that the reflection σ_b across $b^\perp$ lies in S. We show that Φ has the properties of a root system. From $|S| < \infty$ it follows that $|\Phi| < \infty$. Because of $b^\perp = (-b)^\perp$, we have $-\Phi = \Phi$. We order the $b \in \Phi$ lexicographically according to the coefficients with respect to the standard basis of $\mathbb{R}^n$. Let $\Pi := \{b \in \Phi : b > 0\}$ be the set of *positive roots* ($b > 0$ means that the first non-zero component of b is positive). Then $\Phi = \Pi \cup (-\Pi)$ holds. Let $\Delta \subseteq \Phi$ be a minimal subset such that for all $b \in \Pi$ there exist numbers $\lambda_s \geq 0$ with $b = \sum_{s\in\Delta} \lambda_s s$. In the following, let $[.,.]$ be the standard inner product on $\mathbb{R}^n$.

Step 1: $[b,c] \leq 0$ for all distinct $b,c \in \Delta$.
Assume $[b,c] > 0$. For $\mu := 2[b,c] > 0$ we have $\sigma_b(c) = c - \mu b$. Because of $\sigma_b\sigma_c\sigma_b = \sigma_{\sigma_b(c)} \in S$, it follows that $\sigma_b(c) \in \Phi$. First, let $\sigma_b(c) \in \Pi$. Then there exist $\lambda_s \geq 0$ with $\sigma_b(c) = \sum_{s\in\Delta} \lambda_s s$. In the case $\lambda_c < 1$, one obtains

$$(1-\lambda_c)c = \sigma_b(c) + \mu b - \lambda_c c = \mu b + \sum_{s\neq c} \lambda_s s.$$

Now, however, one could remove c from Δ, contradicting the minimality of Δ. Thus $\lambda_c \geq 1$ and

$$(\lambda_c - 1)c + \mu b + \sum_{s\neq c} \lambda_s s = 0.$$

Because of $\mu > 0$, this contradicts the definition of Π. This shows $-\sigma_b(c) \in \Pi$. With $-\sigma_b(c) = \sum_{s\in\Delta} \lambda_s s$ it follows that

$$(\mu - \lambda_b)b = -\sigma_b(c) + c - \lambda_b b = c + \sum_{s\neq b} \lambda_s s.$$

In the case $\lambda_b < \mu$, one could remove b from Δ. Thus $\lambda_b \geq \mu$. But then

$$(\lambda_b - \mu)b + c + \sum_{s\neq b} \lambda_s s = 0$$

would be a contradiction to the definition of Π. Overall, $[b,c] \leq 0$ must hold.

Step 2: Δ is linearly independent.
Let $\sum_{s\in\Delta} \lambda_s s = 0$ with $\lambda_s \in \mathbb{R}$ for $s \in \Delta$. Separating the positive and negative coefficients yields $b := \sum_{\lambda_s\geq 0} \lambda_s s = -\sum_{\lambda_t<0} \lambda_t t$. From Step 1 it follows that

$$0 \leq [b,b] = \sum_{\lambda_s\geq 0} \sum_{\lambda_t<0} \lambda_s(-\lambda_t)[s,t] \leq 0$$

and $b = 0$. Since b is a non-negative linear combination of positive roots, it follows that $\lambda_s = 0$ for all $s \in \Delta$.

Step 3: $S = \langle \sigma_b : b \in \Delta \rangle$.
According to Step 2, every root in Φ can be uniquely written as a linear combination of Δ, where either all coefficients are non-negative or all coefficients are non-positive. Let $T := \langle \sigma_b : b \in \Delta \rangle \leq S$ and $b \in \Pi$. Among all elements in the orbit ${}^T b$, we choose $c = \sum_{s \in \Delta} \lambda_s s \in \Pi$ such that $h(c) := \sum_{s \in \Delta} \lambda_s$ is as small as possible. Assume $c \notin \Delta$. Because of $1 = [c, c] = \sum_{s \in \Delta} \lambda_s [c, s]$, there exists a $t \in \Delta$ with $[c, t] > 0$. We have $\sigma_t(c) = c - 2[c, t]t$. Because $c \in \Pi \setminus \Delta$, c is not a multiple of t. According to Step 2, the representation of $\sigma_t(c)$ with respect to Δ is unique. Because $\Phi = \Pi \cup (-\Pi)$, there can be no coefficients with different signs. This shows $\lambda_t \geq 2[c, t]$ and one has the contradiction $h(\sigma_t(c)) < h(c)$ to the choice of c. Thus $c \in \Delta$ and there exists $t \in T$ with $b = t(c)$. As already noted, it follows that $\sigma_b = t\sigma_c t^{-1} \in T$. Thus $S = \langle \sigma_b : b \in \Pi \rangle = T$ holds.

Step 4: S is a Coxeter group.
Let $\Delta = \{b_1, \ldots, b_n\}$ and $\sigma_i = \sigma_{b_i}$ for $i = 1, \ldots, n$. Let $|\langle \sigma_i \sigma_j \rangle| = m_{ij}$ for $1 \leq i, j \leq n$. According to Step 2, $m_{ij} \geq 2$ for $i \neq j$. If applicable, $\sigma_i \sigma_j$ is a rotation by the angle $\varphi := 2\pi / m_{ij} \leq \pi$ in the plane $\langle b_i, b_j \rangle$. It follows that

$$\cos \varphi = [b_j, \sigma_i \sigma_j(b_j)] = -[b_j, \sigma_i(b_j)] = -[b_j, b_j - 2[b_j, b_i]b_i] = -1 + 2[b_i, b_j]^2,$$
$$[b_i, b_j]^2 = \frac{\cos \varphi + 1}{2} = \frac{\cos(\varphi/2)^2 - \sin(\varphi/2)^2 + 1}{2} = \cos(\varphi/2)^2.$$

From Step 1, one obtains $[b_i, b_j] = -\cos(\varphi/2) = -\cos(\pi/m_{ij})$. Let $G = \langle x_1, \ldots, x_n \rangle$ be the Coxeter group with parameters m_{ij}. With the notation from Definition 10.8, $[.,.]_G$ is now the standard inner product and the monomorphism σ from Theorem 10.9 maps G to S. Because of Step 3, σ is surjective and $S \cong G$ is a Coxeter group. □

Remark 10.42.

(i) G is finite if and only if the matrix $(-\cos(\pi/m_{ij}))_{ij}$ is positive definite. For $G = S_n$ one obtains the matrix

$$\begin{pmatrix} 1 & -1/2 & & 0 \\ -1/2 & \ddots & \ddots & \\ & \ddots & 1 & -1/2 \\ 0 & & -1/2 & 1 \end{pmatrix}.$$

(ii) In the following we call G *positive semidefinite* if $[.,.]_G$ is positive semidefinite. In particular, every finite Coxeter group is positive semidefinite.

Lemma 10.43. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be symmetric, positive semidefinite and indecomposable (i. e. for every partition $\{1, \ldots, n\} = I \dot\cup J$ there exist $i \in I$, $j \in J$ with $a_{ij} \neq 0$). Furthermore, let $a_{ij} \leq 0$ for $i \neq j$. Then*

(i) $\mathrm{Ker}(A) = \{v \in \mathbb{R}^n : vAv^{\mathrm{t}} = 0\}$ *and* $\dim \mathrm{Ker}(A) \leq 1$.

(ii) The eigenspace for the smallest eigenvalue of A is spanned by a positive eigenvector (i. e. all components are positive).

Proof.

(i) Certainly $\operatorname{Ker}(A)$ lies in $N := \{v \in \mathbb{R}^n : vAv^{\mathrm{t}} = 0\}$. Conversely, let $v \in N$. According to the spectral theorem, there exists an orthogonal matrix S with $D := SAS^{\mathrm{t}} = \operatorname{diag}(d_1, \ldots, d_n)$ and $d_1, \ldots, d_n \geq 0$. For $w := vS^{\mathrm{t}}$ it follows that

$$\sum_{i=1}^{n} w_i^2 d_i = wDw^{\mathrm{t}} = vAv^{\mathrm{t}} = 0.$$

Thus $w_i = 0$ or $d_i = 0$ for all i. This shows $0 = S^{\mathrm{t}}Dw^{\mathrm{t}} = Av^{\mathrm{t}}$ and $N \subseteq \operatorname{Ker}(A)$.

Now let $x \in N \setminus \{0\}$ and $z = (|x_1|, \ldots, |x_n|)$. Because $a_{ij} \leq 0$ for $i \neq j$, it holds that

$$0 \leq zAz^{\mathrm{t}} = \sum_{i,j=1}^{n} a_{ij}|x_i x_j| \leq \sum_{i,j=1}^{n} a_{ij} x_i x_j = xAx^{\mathrm{t}} = 0,$$

thus $z \in N$. Let $I := \{1 \leq i \leq n : z_i = 0\}$ and $J = \{1, \ldots, n\} \setminus I$. From $z \in \operatorname{Ker}(A)$ it follows that $\sum_{j \in J} a_{ij} z_j = 0$ for $i = 1, \ldots, n$. Because $z_j > 0$, this is only possible if $a_{ij} = 0$ for all $i \in I$ and $j \in J$. Since A is indecomposable, $I = \varnothing$ must hold, i. e. all components of z are positive and all components of x are non-zero. Since $x \neq 0$ was arbitrary, we conclude $\dim N \leq 1$ (otherwise one could combine a linear combination with a 0-component from two linearly independent vectors).

(ii) Let $d := \min\{d_1, \ldots, d_n\} \geq 0$. Then $B := A - d1_n$ also satisfies the assumptions of the theorem. The claim now follows from the proof of (i) for B. □

Lemma 10.44. *Let G be irreducible and positive semidefinite. We construct a proper subgraph D from $C(G)$ by removing vertices or edges or by reducing edge weights. Then the Coxeter group associated with D is finite.*

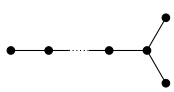
Proof. We number the vertices of $C(G)$ such that D is formed from the first k vertices. Let $A = (a_{ij}) = ([b_i, b_j]_G) \in \mathbb{R}^{k \times k}$ and $B = (b_{ij}) = (-\cos(\pi/m'_{ij})) \in \mathbb{R}^{k \times k}$ be the corresponding matrices. Then $b_{ij} \geq -\cos(\pi/m_{ij}) = a_{ij}$ holds. Suppose B is not positive definite. Let $v \in \mathbb{R}^k \setminus \{0\}$ with $vBv^{\mathrm{t}} \leq 0$ and $w = (|v_1|, \ldots, |v_k|, 0, \ldots, 0) \in \mathbb{R}^n$. Since G is positive semidefinite, it follows that

$$0 \leq wAw^{\mathrm{t}} = \sum_{i,j=1}^{k} a_{ij}|v_i||v_j| \leq \sum_{i,j=1}^{k} b_{ij}|v_i||v_j| \leq \sum_{i,j=1}^{k} b_{ij} v_i v_j = vBv^{\mathrm{t}} \leq 0$$

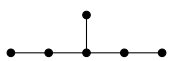
and $wAw^{\mathrm{t}} = 0$. According to Lemma 10.43, all components of w are positive, i. e. $k = n$ and $v_i \neq 0$ for $i = 1, \ldots, k$. From this it follows that $a_{ij} = b_{ij}$. Now, however, D is no longer a *proper* subgraph. Contradiction. □

Theorem 10.45 (Coxeter)**.** *Every finite irreducible Coxeter group G belongs to one of the following families:*[24]

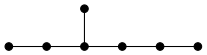
(A_n) *with $n \geq 1$.*

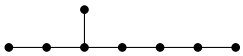
(D_n)  *with $n \geq 4$.*

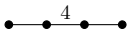
(B_n)  *with $n \geq 2$.*

(E_6) 

[24] These special graphs are called *Dynkin diagrams.*

(E_7) 

(E_8) 

(F_4) 

(H_3) 

(H_4) 

$(I_2(m))$  *with* $m \geq 5$.

Proof.

Existence: Since the given graphs are connected, the corresponding groups are irreducible (Theorem 10.33). We show that they are all finite. All graphs are trees. If one removes a suitable leaf, one obtains a tree that also has one of the given types. We can therefore argue by induction and only need to show that the matrix $M := 2([b_i, b_j])$ has a positive determinant (Sylvester's criterion). For $n = 2$ one obtains

$$\det M = 4(1 - \cos(\pi/m)^2) = 4\sin(\pi/m)^2 > 0$$

because of $m = m_{12} \geq 3$. Now let $n \geq 3$. We number the vertices $e_1, \ldots, e_n$ such that e_n is a leaf and the corresponding edge $\{e_{n-1}, e_n\}$ has weight $m = 3$ or 4. Laplace expansion along the last column shows

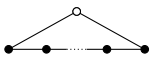
$$\det M = 2\det M_{n-1} - 4\cos(\pi/m)^2 \det M_{n-2} = 2\det M_{n-1} - \lambda \det M_{n-2}$$

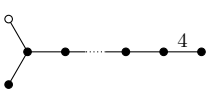
with $\lambda \in \{1, 2\}$, since $\cos(\pi/3) = 1/2$ and $\cos(\pi/4) = 1/\sqrt{2}$. Using $\sin(\pi/5) = \sqrt{10 - 2\sqrt{5}}/4$, one calculates inductively

$$\begin{aligned}
(A_n)&\colon \det M = 2n - (n-1) = n + 1,\\
(B_n)&\colon \det M = 2 \cdot 2 - 2 = 2,\\
(D_n)&\colon \det M = 2 \cdot 4 - 4 = 4,\\
(E_6)&\colon \det M = 2 \cdot 4 - 5 = 3,\\
(E_7)&\colon \det M = 2 \cdot 3 - 4 = 2,\\
(E_8)&\colon \det M = 2 \cdot 2 - 3 = 1,\\
(F_4)&\colon \det M = 2 \cdot 2 - 3 = 1,\\
(H_3)&\colon \det M = 2 \cdot 4\sin(\pi/5)^2 - 2 = \frac{10 - 2\sqrt{5} - 4}{2} = 3 - \sqrt{5} > 0,\\
(H_4)&\colon \det M = 2(3 - \sqrt{5}) - 4\sin(\pi/5)^2 = \frac{12 - 4\sqrt{5} - 5 + \sqrt{5}}{2} = \frac{7 - 3\sqrt{5}}{2} > 0.
\end{aligned}$$

Auxiliary graphs: We add another vertex to some of the graphs and obtain the following Coxeter graphs[25] (the number of vertices is now $n + 1$):

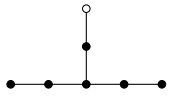
$(\tilde{A}_1)$ 

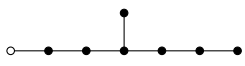
$(\tilde{A}_n)$  with $n \geq 2$.

$(\tilde{B}_n)$  with $n \geq 3$.

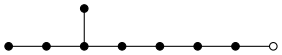
$(\tilde{C}_n)$  with $n \geq 2$.

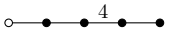
$(\tilde{D}_n)$  with $n \geq 4$.

$(\tilde{E}_6)$ 

$(\tilde{E}_7)$ 

[25] The designations are not uniform in the literature.

$(\tilde{E}_8)$ 

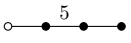
$(\tilde{F}_4)$ 

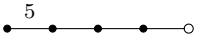
$(\tilde{G}_2)$ 

We show that the corresponding bilinear form is positive semidefinite, but not positive definite. For $\tilde{A}_n$, the row sums of M are all $2-1-1=0$, i.e., 0 is an eigenvalue and $\det M = 0$. In all other cases, one can apply the above recursion formula (note $\sin(\pi/6) = 1/2$):

$$
\begin{aligned}
(\tilde{B}_n)&: \det M = 2 \cdot 2 - 2 \cdot 2 = 0,\\
(\tilde{C}_n)&: \det M = 2 \cdot 2 - 2 \cdot 2 = 0,\\
(\tilde{D}_n)&: \det M = 2 \cdot 4 - 2 \cdot 4 = 0,\\
(\tilde{E}_6)&: \det M = 2 \cdot 3 - 6 = 0,\\
(\tilde{E}_7)&: \det M = 2 \cdot 2 - 4 = 0,\\
(\tilde{E}_8)&: \det M = 2 \cdot 1 - 2 = 0,\\
(\tilde{F}_4)&: \det M = 2 \cdot 1 - 2 = 0,\\
(\tilde{G}_2)&: \det M = 2 \cdot 1 - 2 = 0.
\end{aligned}
$$

Adding a vertex to H_3 and H_4, one obtains the following Coxeter graphs:

(Z_4): 

(Z_5): 

Because of

$$
\begin{aligned}
(Z_4)&: \det M = 2(3-\sqrt{5}) - 3 = 3 - 2\sqrt{5} < 0,\\
(Z_5)&: \det M = 7 - 3\sqrt{5} - (3-\sqrt{5}) = 4 - 2\sqrt{5} < 0
\end{aligned}
$$

the corresponding bilinear forms are no longer positive semidefinite.

Uniqueness: Now let G be a finite irreducible Coxeter group of rank n. Let m be the largest edge weight. Suppose that $C(G)$ does not belong to the positive (semi)definite graphs described above.

(1) Since all connected graphs with $n = 2$ have already been listed, $n \geq 3$ holds.

(2) According to Lemma 10.44, $\tilde{A}_1$ is not a subgraph of $C(G)$ and therefore $m < \infty$.

(3) Since $\tilde{A}_k$ is not a subgraph, $C(G)$ must be a tree. Let us first assume $m = 3$.

(4) Because $C(G) \neq A_n$, $C(G)$ has a branching point.

(5) Since $\tilde{D}_k$ is not a subgraph, there is exactly one branching point e.

(6) Since $\tilde{D}_4$ is not a subgraph, e has exactly three branches. Let $a \leq b \leq c$ be the number of vertices of the three branches.

(7) Since $\tilde{E}_6$ is not a subgraph, $a = 1$.

(8) Since $\tilde{E}_7$ is not a subgraph, $b \leq 2$.

(9) Because $C(G) \neq D_n$, $b = 2$.

(10) Since $\tilde{E}_8$ is not a subgraph, $c \leq 4$.

(11) Since $C(G)$ is neither E_6, E_7 nor E_8, the case $m = 3$ cannot occur. So let $m \geq 4$.

(12) Since $\tilde{C}_k$ is not a subgraph, there can only be one edge with weight > 3.

(13) Since $\tilde{B}_k$ is not a subgraph, there are no branching points, i.e., $C(G)$ is a line. Now let us assume $m = 4$.

(14) Because $C(G) \neq B_n$, the two outer edges have weight 3.

(15) Since $\tilde{F}_4$ is not a subgraph, $n = 4$ must hold.

(16) Because $C(G) \neq F_4$, $m = 4$ cannot hold. So let $m \geq 5$.

(17) Since $\tilde{G}_2$ is not a subgraph, $m = 5$ holds.

(18) Since Z_4 is not a subgraph, an outer edge must have weight m.

(19) Since Z_5 is not a subgraph, $n \leq 4$ must hold.

(20) But then $C(G)$ would be equal to H_3 or H_4. Contradiction. □

Theorem 10.46. *For the groups in Theorem 10.45, the following holds:*

(A_n) $G \cong S_{n+1}$

(B_n) $G \cong C_2 \wr S_n$.

(D_n) $G \cong C_2^{n-1} \rtimes S_n$.

(E_6) $G \cong \mathrm{Aut}(\mathrm{SU}(4,2)) \cong \mathrm{SU}(4,2) \rtimes C_2$ *with* $|G| = 51.840 = 2^7 3^4 5$.

(E_7) $G \cong \mathrm{Sp}(6,2) \times C_2$ *with* $|G| = 2.903.040 = 2^{10} 3^4 5 \cdot 7$.

(E_8) $G = 2.\Omega^+(8,2).2$ *with* $|G| = 696.729.600 = 2^{14} 3^5 5^2 7$.

(F_4) $G \cong (C_2^3 \rtimes S_4) \rtimes S_3$.

(H_3) $G \cong A_5 \times C_2$.

(H_4) $|G| = 14.400 = 2^6 3^2 5^2$.

($I_2(m)$) $G \cong D_{2m}$.

Sketch of proof. For (A_n) and ($I_2(m)$), the claim is known from Example 10.32. Since $[.,.]_G$ is positive definite, $[.,.]_G$ coincides with the standard scalar product on $\mathbb{R}^n$ up to a choice of basis. We choose concrete vectors $b_1, \ldots, b_n$ with the given values $[b_i, b_j]$ and determine $\sigma(G)$ explicitly. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$.

(B_n) Define $b_i := \frac{1}{\sqrt{2}}(e_i - e_{i+1})$ for $i = 1, \ldots, n-1$ and $b_n = e_n$. It holds that $[b_i, b_i] = 1$, $[b_i, b_j] = 0$ for $|j - i| > 1$, $[b_i, b_{i+1}] = -1/2 = -\cos(\pi/3)$ for $i = 1, \ldots, n-2$ and $[b_{n-1}, b_n] = -1/\sqrt{2} = -\cos(\pi/4)$ as desired. It holds that $b_i^\perp = \langle e_j, e_i + e_{i+1} : j \neq i \rangle$. With respect to $e_1, \ldots, e_n$, $\sigma(x_i)$ corresponds to the permutation matrix for the transposition $(i, i+1)$ for $i = 1, \ldots, n-1$ and $\sigma(x_n) = \mathrm{diag}(1, \ldots, 1, -1)$. Therefore, $\sigma(G)$ is the group of signed permutation matrices. This shows $G \cong C_2 \wr S_n$.

(D_n) Define $b_i := \frac{1}{\sqrt{2}}(e_i - e_{i+1})$ for $i = 1, \ldots, n-1$ and $b_n = \frac{1}{\sqrt{2}}(e_{n-1} + e_n)$. It holds that $[b_{n-2}, b_{n-1}] = [b_{n-2}, b_n] = -1/2$ and $[b_{n-1}, b_n] = 0$. Again, $\sigma(x_i)$ for $i < n$ are permutation matrices and

$$\sigma(x_n) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & -1 & 0 \end{pmatrix}.$$

Therefore, $\sigma(G)$ consists of the signed permutation matrices with an even number of -1 entries. The corresponding diagonal matrices form a normal subgroup C_2^{n-1} with complement S_n in G.

(H_3) For $x := x_3x_2$, $y := x_2x_1$, it holds that $|\langle x\rangle| = 3$, $|\langle y\rangle| = 5$ and $|\langle xy\rangle| = 2$. According to Example 2.16, there exists an epimorphism $A_5 \to \langle x, y\rangle =: H$. Since A_5 is simple, it follows that $H \cong A_5$. From $\sigma(H) \leq \mathrm{SL}(3, \mathbb{R})$ and $\det(\sigma_1) = -1$, it follows that $|G| = 2|H| = 120$. According to Remark 6.3, $G \in \{S_5, A_5 \times C_2\}$ holds. In the case $G \cong S_5$, we would have $D_{10} \cong \langle x_1, x_2\rangle \leq A_5$ in contradiction to $\det(x_1) = -1$.

The other cases are quite elaborate and can be handled with GAP:

```
for para in [["E",6],["E",7],["E",8],["F",4]] do
  L:=SimpleLieAlgebra(para[1],para[2],Rationals);
  R:=RootSystem(L);;
  W:=WeylGroup(R);
  Print(para," ",Order(W),"\n");
od;
F:=FreeGroup("a","b","c","d");;
AssignGeneratorVariables(F);;
H4:=F/[a^2,b^2,c^2,d^2,(a*b)^5,(b*c)^3,(c*d)^3,Comm(a,c),Comm(a,d),Comm(b,d)];
    #Comm(a,b)=[a,b]
Size(H4);
```

□

Remark 10.47.

(i) The matrix -1_n clearly lies in the center of the Coxeter group $G \cong C_2^n \rtimes S_n$ of type (B_n) (signed permutation matrices). According to Theorem 10.36, $\mathrm{Z}(G) = \langle -1_n\rangle$. If n is odd, then $G = \mathrm{Z}(G) \times H$, where H is isomorphic to the Coxeter group of type (D_n) (permutation matrices with an even number of -1 entries). Nevertheless, G is irreducible. For $n = 3$ one obtains

$$G \cong C_2 \times (C_2^2 \rtimes S_3) \cong C_2 \times S_4.$$

This group permutes the eight vertices of the cube $(\pm 1, \pm 1, \pm 1)$ and is therefore its symmetry group (and that of the octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$). For $n \geq 4$ one obtains the symmetry group of the n-dimensional hypercube. The Coxeter group $H \cong C_2^2 \rtimes S_3 \cong S_4$ of type (D_3) permutes the four vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ of the tetrahedron and is therefore its symmetry group. As is well known, S_4 is also the Coxeter group of type (A_3).

(ii) One can show that the Coxeter group of type (H_3) is the symmetry group of the dodecahedron (or the icosahedron). The Coxeter group of type (F_4) is an extension of the group of type (D_4) by S_3. The Coxeter group of type (E_7) is a Schur extension of the orthogonal group $GO^+(8, 2) \cong \Omega^+(8, 2).2$ with center C_2.

(iii) A (complex) *Lie algebra* is a finite-dimensional $\mathbb{C}$-vector space L with an alternating bilinear map $L \times L \to L$, $(v, w) \mapsto [v, w]$, which satisfies the *Jacobi identity*

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

fulfilled. For example $L = \mathbb{C}^{n\times n}$ with $[v, w] := vw - wv$. One calls L *simple* if $[L, L] \neq 0$ and if no subspace $0 < U < L$ with $[L, U] \subseteq U$ exists. The classification of simple Lie algebras leads to the diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2.[26] These correspond more or less to the simple groups of Lie type (for example, A_n corresponds to the family $\mathrm{PSL}(n + 1, q)$ and C_n to the family $\mathrm{PSp}(2n, q)$).

[26] See Algebra notes

(iv) One calls $\sigma \in \mathrm{GL}(\mathbb{C}^d)$ a *complex reflection* if $|\langle\sigma\rangle| < \infty$ and $\dim \mathrm{Ker}(\sigma - \mathrm{id}) = d - 1$. A *complex reflection group* is a finite subgroup $G \leq \mathrm{GL}(\mathbb{C}^d)$ generated by complex reflections. Every reflection group is clearly also a complex reflection group. Shephard and Todd have completely classified the irreducible complex reflection groups. Apart from 34 exceptions, every such group has the form

$$G(m,d,n) := \big\langle (x_1,\dots,x_n;\sigma) \in C_m \wr S_n : (x_1 \dots x_n)^{m/d} = 1 \big\rangle,$$

where $n, m, d \in \mathbb{N}$ and $d \mid m$. Clearly $G(m,1,1) \cong C_m$, $G(1,1,n) \cong S_n$, $G(2,1,n) \cong C_2 \wr S_n$ (type (B_n)), $G(2,2,n)$ type (D_n) and $G(m,m,2) \cong D_{2m}$ (type $(I_2(m))$). The 34 exceptional groups are unfortunately only available in the old GAP version 3.[27]

Definition 10.48. For $a, b, c \in \mathbb{Z} \setminus \{0\}$,

$$D(a,b,c) := \langle x, y \mid x^a = y^b = (xy)^c = 1 \rangle$$

is called a *von Dyck group*.

Lemma 10.49. *$D(a,b,c)$ does not depend on the order and the signs of a, b, c.*

Proof. The independence from the sign is obvious. So let $a, b, c \geq 1$. Because of $(xy)^c = 1 \iff (yx)^c = 1$, one can swap a and b. Now let $x' := xy$ and $y' := y^{-1}$. Then $x^a = y^b = (xy)^c = 1$ is equivalent to $(x')^c = (y')^b = (x'y')^a = 1$. Thus, one can also swap a and c. □

Theorem 10.50. *$D(a,b,c)$ is finite if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. If applicable, one of the following cases occurs:*

(i) $D(1,b,c) \cong C_{\gcd(b,c)}$.

(ii) $D(2,2,c) \cong D_{2c}$.

(iii) $D(2,3,3) \cong A_4$.

(iv) $D(2,3,4) \cong S_4$.

(v) $D(2,3,5) \cong A_5$.

Proof. According to Lemma 10.49, we can assume $1 \leq a \leq b \leq c$. First, let $a = 1$. By Euclid, there exist $\alpha, \beta \in \mathbb{Z}$ with $\gcd(b,c) = \alpha b + \beta c$. Thus $y^{\gcd(b,c)} = (y^b)^\alpha + (y^c)^\beta = 1$ in $D(1,b,c)$. It follows that $|D(1,b,c)| \leq \gcd(b,c)$. Conversely, $C_{\gcd(b,c)}$ also satisfies the relations of $D(1,b,c)$. Thus (i) holds.

In the case $a = b = 2$, the claim follows from Example 1.17. The case $(a,b,c) = (2,3,3)$ was treated in Exercise 2.

Let $(a,b,c) = (2,3,4)$ and $G := \langle x, y \mid x^4 = y^2 = (xy)^3 = 1 \rangle \cong D(2,3,4)$ and $H := \langle x \rangle \leq G$. We consider the cosets

$$H, yH, xyH, x^2yH, x^3yH, yx^2yH.$$

Because of $xyx = yx^{-1}y$ and $xyx^2y = yx^{-1}yxy = yx^2xyxy = yx^2yx^{-1}$, the cosets are permuted by left multiplication by x. Because of $yxy = x^{-1}yx^{-1} = x^3yx^3$, y also permutes these cosets and we see that every element in G is contained in one of these cosets. Thus $|G| \leq |G : H||H| \leq 24$. On the other hand, one sees that $x' := (1,2,3,4)$ and $y' := (1,2)$ in S_4 also satisfy the relations of G. This yields

[27] `https://webusers.imj-prg.fr/~jean.michel/gap3/`

(iv). The case $(a,b,c) = (2,3,5)$ has already been treated several times (Remark 1.18, Example 2.16). It is easy to see that in all other cases $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$ holds.

Let

$$H := \langle u, v, w \mid u^2 = v^2 = w^2 = (uv)^a = (vw)^b = (uw)^c = 1 \rangle$$

be a Coxeter group. Obviously $N := \langle uv, vw \rangle \leq H$ with $|H : N| = 2$. Furthermore, there exists an epimorphism $D(a,b,c) \to N$. It therefore suffices to show that H is infinite for $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$. In the case $a \geq 3$, $C(H)$ is a cycle and $|H| = \infty$ according to Theorem 10.45. So let $a = 2$ and $\frac{1}{b} + \frac{1}{c} \leq \frac{1}{2}$. Then $b, c \geq 3$ and H is irreducible. For $b = 3$, we have $c \geq 6$ and the claim follows in any case from Theorem 10.45. □

Remark 10.51. A finite group $G \neq 1$ is called a *Hurwitz group*, if $G = \langle x, y \rangle$ with $x^2 = y^3 = (xy)^7 = 1$ holds. These are thus exactly the finite non-trivial factor groups of the infinite von Dyck group $D(2,3,7)$. According to Exercise 37, all Hurwitz groups are perfect and indeed many simple groups, among others A_n for $n \geq 168$, $\mathrm{GL}(3,2)$ and the sporadic *Monster group* are Hurwitz groups.

Theorem 10.52. *For $n \in \mathbb{N}$, there are infinitely many prime numbers $p \equiv 1 \pmod{n}$.*[28]

Proof. Let $p_1, \ldots, p_s$ be prime numbers with $p_i \equiv 1 \pmod{n}$ for $i = 1, \ldots, s$ (the case $s = 0$ is allowed). Let $m := np_1 \ldots p_s$. The cyclotomic polynomial Φ_m induces, as a monic polynomial, an unbounded function $\mathbb{R} \to \mathbb{R}$. Therefore, there exists a $k \in \mathbb{N}$ with $\Phi_m(km) > 1$. Let p be a prime divisor of $\Phi_m(km)$. Because of $\Phi_m(km) \mid ((km)^m - 1)$, we have $p \nmid m$ and the order r of $mk + p\mathbb{Z} \in \mathbb{F}_p^\times$ divides $\gcd(m, p-1)$. Assume $r < m$ and set $a := (mk)^r \equiv 1 \pmod{p}$. Then

$$\frac{m}{r} \equiv 1 + a + \ldots + a^{\frac{m}{r}-1} = \frac{a^{\frac{m}{r}} - 1}{a - 1} = \frac{(mk)^n - 1}{(mk)^r - 1} = \prod_{\substack{d \mid m \\ d \nmid r}} \Phi_d(mk) \equiv 0 \pmod{p}$$

holds in contradiction to $p \nmid m$. Thus $r = m$ holds and $m \mid p - 1$, i. e. $p \equiv 1 \pmod{m}$. Because of $n \mid m$, we also have $p \equiv 1 \pmod{n}$. On the other hand, $m = np_1 \ldots p_s$ and $p \notin \{p_1, \ldots, p_s\}$. We have thus found a new prime number in the residue class $1 + n\mathbb{Z}$. □

Theorem 10.53 (Miller)**.** *For $a, b, c \in \mathbb{N} \setminus \{1\}$, there exists a finite group $G = \langle x, y \rangle$ with $|\langle x \rangle| = a$, $|\langle y \rangle| = b$ and $|\langle xy \rangle| = c$.*

Proof (Holt). Instead of constructing G as a quotient of a von Dyck group, we give an independent proof. Theorem 10.52 guarantees the existence of a prime $p \equiv 1 \pmod{2abc}$. Let $\zeta_a, \zeta_b, \zeta_c \in \mathbb{F}_p^\times$ be elements with order $2a, 2b$ and $2c$ respectively. Let $\lambda \in \mathbb{F}_p$ be initially arbitrary. Then

$$x := \begin{pmatrix} \zeta_a & 1 \\ 0 & \zeta_a^{-1} \end{pmatrix} \in \mathrm{SL}(2,p), \qquad y := \begin{pmatrix} \zeta_b & 0 \\ \lambda & \zeta_b^{-1} \end{pmatrix} \in \mathrm{SL}(2,p)$$

have the eigenvalues $\zeta_a \neq \zeta_a^{-1}$ and $\zeta_b \neq \zeta_b^{-1}$ respectively. In particular, x, y are diagonalizable and it follows that $|\langle x \rangle| = 2a$ and $|\langle y \rangle| = 2b$. Because of $\mathrm{tr}(xy) = 2\zeta_a\zeta_b + \lambda$, one can choose λ such that $\mathrm{tr}(xy) = \zeta_c + \zeta_c^{-1}$ holds. For the eigenvalues e, f of xy (in a splitting field), $e + f = \mathrm{tr}(xy)$ and $ef = \det(xy) = \det(x)\det(y) = 1$ holds. This shows $\{e, f\} = \{\zeta_c, \zeta_c^{-1}\}$ and xy is also diagonalizable with order $2c$. Since x^a, x^b and $(xy)^c$ have the double eigenvalue -1, $x^a = y^b = (xy)^c = -1_2 \in Z := \mathrm{Z}(\mathrm{SL}(2,p))$ holds. With $\overline{x} := xZ$ and $\overline{y} := yZ$, $G = \langle \overline{x}, \overline{y} \rangle \leq \mathrm{PSL}(2,p)$ satisfies the claim. □

[28] This is a special case of Dirichlet's theorem on arithmetic progressions: The prime numbers are distributed uniformly among the coprime residue classes modulo n, i. e. for $\gcd(k,n) = 1$, the proportion of prime numbers in $k + n\mathbb{Z}$ is exactly $1/\varphi(n)$.

11 Free Products and Amalgams

Example 11.1. The direct product $G \times H$ of two groups G and H is the largest group D with the following properties:

- $G, H \leq D$ and $D = \langle G, H \rangle$,
- $xy = yx$ for all $x \in G$ and $y \in H$.

We construct the largest group that satisfies only the first property.

Definition 11.2. Let $G_I := \{G_i : i \in I\}$ be a non-empty family of groups. A *free product* of G_I is a group G with homomorphisms $\lambda_i \colon G_i \to G$ $(i \in I)$ with the following universal property: For every group H and homomorphisms $\rho_i \colon G_i \to H$, there exists exactly one homomorphism $\varphi \colon G \to H$ with $\rho_i = \varphi\lambda_i$ for all $i \in I$.

Lemma 11.3.

(i) The homomorphisms λ_i are injective and $G = \langle \lambda_i(G_i) : i \in I \rangle$.

(ii) Up to isomorphism, there is at most one free product of G_I.

Proof.

(i) Let $H := G_i$, $\rho_i := \mathrm{id}_H$ and $\rho_j := 1$ for $j \neq i$. Then there exists a $\varphi \colon G \to H$ with $\mathrm{id}_H = \rho_i = \varphi\lambda_i$. In particular, λ_i is injective. Now let $H := \langle \lambda_i(G_i) : i \in I \rangle \leq G$ and $\rho_i := \lambda_i \colon G_i \to H$. Then there exists exactly one $\varphi \colon G \to H$ with $\lambda_i = \varphi\lambda_i$ for $i \in I$. For the same reason, there exists exactly one homomorphism $\varphi' \colon G \to G$ with $\lambda_i = \varphi'\lambda_i$. Obviously, $\varphi' = \mathrm{id}_G$ must hold. If one interprets φ as a map $G \to G$, then $\varphi = \varphi'$ also holds. This shows $G = H$.

(ii) Let H also be a free product of G_I with homomorphisms $\mu_i \colon G_i \to H$. Then there exist homomorphisms $\varphi \colon G \to H$ and $\psi \colon H \to G$ with $\mu_i = \varphi\lambda_i$ and $\lambda_i = \psi\mu_i$ for $i \in I$. It follows that $\psi\varphi\lambda_i = \psi\mu_i = \lambda_i$ and $\varphi\psi\mu_i = \varphi\lambda_i = \mu_i$. Now, as in (i), id_G and id_H are the only homomorphisms with $\mathrm{id}_G\, \lambda_i = \lambda_i$ and $\mathrm{id}_H\, \mu_i = \mu_i$. This shows $\varphi\psi = \mathrm{id}_H$ and $\psi\varphi = \mathrm{id}_G$. In particular, $\varphi \colon G \to H$ is an isomorphism. □

Remark 11.4. According to Lemma 11.3, one can speak of *the* free product of G_I without mentioning the homomorphisms λ_i. One writes $G = \mathrm{Fr}_{i \in I} G_i$ or $G_1 * \ldots * G_n$ if $I = \{1, \ldots, n\}$.[29] Since the λ_i are injective, one can assume $G_i \leq G$ (this corresponds to the formal difference between direct product and direct sum).

Theorem 11.5. *For every non-empty family of groups G_I, there exists $\mathrm{Fr}_{i \in I} G_i$.*

Proof. We modify Definition 11.2. Wlog. let $G_i \cap G_j = \varnothing$ for $i \neq j$. Let W be the set of all formal words of the form $w = g_1 \ldots g_n$ with $n \in \mathbb{N}_0$ and $g_1, \ldots, g_n \in \bigcup_{i \in I} G_i$. One calls w *reduced*, if $g_1, \ldots, g_n \neq 1$ and $\{g_i, g_{i+1}\} \not\subseteq G_j$ for $i = 1, \ldots, n-1$ and all $j \in I$. Obviously, w can always be reduced. Words $v, w \in W$ are called *equivalent*, if they can be reduced to the same word. Let $G = \{[w] : w \in W\}$ be the set of equivalence classes of this relation. Obviously, G is a group with respect to concatenation. We define $\lambda_i \colon G_i \to G$ by $\lambda_i(g) := [g]$ for $g \in G$. Now let H be another group and $\rho_i \colon G_i \to H$ homomorphisms for $i \in I$. Obviously, the map $\varphi \colon G \to H$ with $[w] \mapsto \rho_1(g_1) \ldots \rho_n(g_n)$ is a well-defined

[29] Attention: Danger of confusion with the central product.

homomorphism with $\rho_i = \varphi\lambda_i$ for $i \in I$. Since G is generated by the elements $[g]$ with $g \in \bigcup G_i$ by construction, φ is uniquely determined by ρ_i. □

Lemma 11.6. *Every element in* $\operatorname{Fr}_{i\in I} G_i$ *can be uniquely written as a reduced word.*

Proof. We proceed as in Lemma 1.7. Let R be the set of reduced words $r = g_1 \dots g_n$ with $g_1, \dots, g_n \in \bigcup G_i$. For $g \in G_j$ let

$$ {}^g r := \begin{cases} r & \text{if } g = 1, \\ gg_1 \dots g_n & \text{if } g_1 \notin G_j, \\ g_2 \dots g_n & \text{if } g = g_1^{-1}, \\ (gg_1)g_2 \dots g_n & \text{otherwise.} \end{cases} $$

One easily sees that this describes an action $G_j \to \operatorname{Sym}(R)$. By the universal property, the action can be extended to $G \to \operatorname{Sym}(R)$. For $v, w \in R$ with $[v] = [w]$, it now holds that $v = {}^{[v]}1 = {}^{[w]}1 = w$. □

Remark 11.7. In the following, we will replace $[w]$ by w. Every element in $\operatorname{Fr} G_i$ can then be uniquely written in the reduced form $g_1 \dots g_n$. Conversely, if one has $H = \langle G_i : i \in I \rangle$ such that every element in H can be uniquely written in reduced form, then it follows that $H \cong \operatorname{Fr}_{i\in I} G_i$, because the inclusions $\sigma_i \colon G_i \hookrightarrow H$ can be extended to an isomorphism.

Example 11.8.

(a) A free product of free groups is free: Let $F_i = F_{X_i}$ with $X_i \cap X_j = \varnothing$ for $i \neq j$. Let $X := \bigcup_{i\in I} X_i$ and $F := F_X$. Let $\rho_i \colon F_i \to H$ be homomorphisms. Then there exists exactly one homomorphism $\varphi \colon F \to H$ with $\varphi(x) := \rho_i(x)$ for $x \in X_i$. Because of $F_i = \langle X_i \rangle$, we have $\rho_i = \varphi_{|F_i}$. As a special case, one obtains $F_n = \mathbb{Z} * \ldots * \mathbb{Z}$.

(b) Let $G = \langle x \rangle * \langle y \rangle \cong C_2 * C_2$ and $z := xy \in G$. Because of $z^{-1} = yx$, every element in G can be uniquely written in the form $z^a x^b$ with $a \in \mathbb{Z}$ and $b \in \{0, 1\}$. Because of $xzx^{-1} = z^{-1}$, we have $G = \langle z \rangle \rtimes \langle x \rangle \cong \mathbb{Z} \rtimes C_2 \cong D_\infty$.

Lemma 11.9. *It holds that* $\operatorname{PSL}(2, \mathbb{Z}) \cong C_2 * C_3$.

Proof. Let $A := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ in $\operatorname{SL}(2, \mathbb{Z})$. We first show $\operatorname{SL}(2, \mathbb{Z}) = \langle A, B \rangle$. Suppose indirectly that $C = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{SL}(2, \mathbb{Z}) \setminus \langle A, B \rangle$ with $|a| + |c|$ minimal. Assume $a \neq 0 \neq c$. Then

$$ (AB)^s C = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + sc & b + sd \\ c & d \end{pmatrix} \notin \langle A, B \rangle, $$

$$ (BA)^{-r} C = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ ra + c & rb + d \end{pmatrix}. $$

In the case $|a| \geq |c|$, one can choose s such that $|a + sc| + |c| < |a| + |c|$ holds. Otherwise, one can choose r such that $|a| + |ra + c| < |a| + |c|$ holds. Contradiction. Thus $a = 0$ or $c = 0$. In the first case, it would be

$$ C = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & d \end{pmatrix} \in \{BA^2(AB)^{-d-1}, B(AB)^{d-1}\} $$

and in the second case, it would be

$$ C = \begin{pmatrix} \pm 1 & b \\ 0 & \mp 1 \end{pmatrix} \in \{(AB)^b, A^2(AB)^{-b}\}. $$

This shows $\mathrm{SL}(2,\mathbb{Z}) = \langle A, B\rangle$.

Because of $A^2 = -1 = B^3$, $\overline{A}, \overline{B} \in \mathrm{PSL}(2,\mathbb{Z})$ have order 2 and 3 respectively (note $\mathrm{Z}(\mathrm{SL}(2,\mathbb{Z})) = \langle -1_2\rangle$). Let $G := \langle x\rangle * \langle y\rangle \cong C_2 * C_3$. Then there exists an epimorphism $\varphi\colon G \to \mathrm{PSL}(2,\mathbb{Z})$ with $\varphi(x) = \overline{A}$ and $\varphi(y) = \overline{B}$. Suppose there exists $w \in \mathrm{Ker}(\varphi) \setminus \{1\}$. Then w is an alternating product of x and $y^{\pm 1}$. After conjugation, we can assume $w = xy^{\epsilon_1} \dots xy^{\epsilon_n}$ with $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$. The matrices AB and $AB^{-1} = \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$ have non-negative and non-positive entries, respectively. The same property would also have to hold for

$$\overline{AB^{\epsilon_1} \dots AB^{\epsilon_{n-1}} AB^{-\epsilon_n}} = \varphi(wy^{\epsilon_n}) = \varphi(w)\varphi(y^{\epsilon_n}) = \overline{B^{\epsilon_n}}.$$

In fact, however, B^{ϵ_n} has both positive and negative entries. Thus φ is an isomorphism. □

Lemma 11.10. *Let $F := \mathrm{Fr}_{i\in I} G_i$ with $G_i \neq 1$ for all $i \in I$. Then:*

(i) Every element $g \in F$ of finite order lies, up to conjugation, in some G_i.

(ii) For $g \in G_i \setminus \{1\}$, $\mathrm{C}_F(g) = \mathrm{C}_{G_i}(g)$ holds.

(iii) For $|I| \geq 2$, $|F| = \infty$ and $\mathrm{Z}(F) = 1$ hold.

Proof.

(i) Let $g = g_1 \dots g_n \in F$ be reduced with $n \geq 2$. After conjugation, we can assume that g_1 and g_n lie in different factors G_i. Obviously, $g^k \neq 1$ for all $k \geq 1$.

(ii) Let $x = x_1 \dots x_n \in \mathrm{C}_F(g)$ be reduced and wlog. $x_n \notin G_i$. Then xgx^{-1} would also be reduced and $xgx^{-1} \neq g$.

(iii) For $g \in G_i \setminus \{1\}$ and $h \in G_j \setminus \{1\}$ with $i \neq j$, gh is reduced with infinite order. Furthermore, $\mathrm{Z}(F) \leq \mathrm{C}_F(g) \cap \mathrm{C}_F(h) \leq G_i \cap G_j = 1$ by (ii). □

Theorem 11.11. *Let $G = \mathrm{Fr}_{i\in I} G_i$ and $H = \mathrm{Fr}_{i\in I} H_i$ be free products over the same index set I. Let $\sigma_i\colon G_i \to H_i$ be homomorphisms for $i \in I$. Then there exists exactly one homomorphism $\varphi\colon G \to H$ that extends all σ_i. Furthermore, $\mathrm{Ker}(\varphi)$ is the normal closure of $\bigcup_{i\in I} \mathrm{Ker}(\sigma_i)$ in G.*

Proof. The existence and uniqueness of φ follows from the universal property. Let $N \trianglelefteq G$ be the normal closure of $\bigcup_{i\in I} \mathrm{Ker}(\sigma_i)$ in G. Certainly $N \subseteq \mathrm{Ker}(\varphi)$. Assume $\mathrm{Ker}(\varphi) \nsubseteq N$ and choose $g = g_1 \dots g_n \in \mathrm{Ker}(\varphi) \setminus N$ reduced with minimal n and $g_k \in G_{i_k}$ for $k = 1, \dots, n$. If $\sigma_{i_k}(g_k) \neq 1$ for $k = 1, \dots, n$, then $1 = \varphi(g) = \sigma_{i_1}(g_1) \dots \sigma_{i_n}(g_n)$ would be reduced in H. This contradiction shows $\sigma_{i_k}(g_k) = 1$ for some $1 \leq k \leq n$. But then $g_1 \dots g_{k-1} g_{k+1} \dots g_n \in \mathrm{Ker}(\varphi) \setminus N$ in contradiction to the choice of n. □

Corollay 11.12. *Let $G_i = \langle X_i \mid R_i\rangle$ be presentations for $i \in I$. Then*

$$\mathrm{Fr}_{i\in I} G_i \cong \langle \dot{\bigcup}_{i\in I} X_i \mid \dot{\bigcup}_{i\in I} R_i\rangle.$$

Proof. Let $F_i := F_{X_i}$ and F be the free group with respect to $\dot{\bigcup}_{i\in I} X_i$. By Example 11.8, $F = \mathrm{Fr}_{i\in I} F_i$. Let $\sigma_i\colon F_i \to G_i$ be the canonical epimorphism. By Theorem 11.11, there exists an epimorphism $\varphi\colon F \to \mathrm{Fr}_{i\in I} G_i$ with $\mathrm{Ker}(\varphi) = \langle \bigcup \mathrm{Ker}(\sigma_i)\rangle^F = \langle \bigcup R_i^{F_i}\rangle^F = \langle \bigcup R_i\rangle^F$. This shows the claim. □

Remark 11.13.

(i) KUROSH has shown that every subgroup H of $F := \mathrm{Fr}_{i\in I} G_i$ has the form

$$F_0 * \mathrm{Fr}_{i\in I} \mathrm{Fr}_{HxG_i\in H\backslash F/G_i}(H \cap xG_ix^{-1})$$

where F_0 is free (here $H\backslash F/G_i$ is the set of *double cosets* with respect to H and G_i).

(ii) From direct products one can form central products by quotienting out central subgroups. We construct a non-commutative variant.

Definition 11.14. Let $G_I = \{G_i : i \in I\}$ be a family of groups. Let H be a group and $\sigma_i \colon H \to G_i$ monomorphisms for $i \in I$. Let N be the normal closure of $\{\sigma_i(h)^{-1}\sigma_j(h) : h \in H, i, j \in I\}$ in $F := \mathrm{Fr}_{i\in I} G_i$. One calls F/N the *amalgam of G_I w.r.t. H*.

Remark 11.15. Because of $\sigma_i(h) \equiv \sigma_j(h) \pmod N$, it holds that $\sigma_i(H)N/N = \sigma_j(H)N/N \leq F/N$. Thus, one identifies the isomorphic copies of H in G_i. In the case $H = 1$, $N = 1$ and one obtains the ordinary free product. In contrast to the free product, F/N also depends on the choice of the monomorphisms σ_i.

Example 11.16. We had proven $\mathrm{PSL}(2,\mathbb{Z}) = \langle \overline{A}, \overline{B}\rangle \cong C_2 * C_3$ in Lemma 11.9. One easily sees that $\mathrm{SL}(2,\mathbb{Z}) = \langle A, B\rangle$ is an amalgam of $\langle A\rangle = C_4$ and $\langle B\rangle \cong C_6$ w.r.t. $\langle A^2\rangle = \langle B^3\rangle = \langle -1_2\rangle = \mathrm{Z}(\mathrm{SL}(2,\mathbb{Z}))$.

Lemma 11.17. *In the situation of Definition 11.14, let R_i be a transversal?! for the cosets of $\sigma_i(H)$ in G_i with $1 \in R_i$. Then every element in F/N can be uniquely written in the reduced form $r_1 \dots r_n hN$ with $n \in \mathbb{N}_0$, $r_k \in R_{i_k} \setminus \{1\}$, $i_k \neq i_{k+1}$ and $h \in \sigma_{i_n}(H)$.*

Proof. Let $g = g_1 \dots g_n N \in F/N$ with $g_k \in G_{i_k}$ be arbitrary. Then there exist $r_1 = \overline{g_1} \in R_{i_1}$ and $h_1 \in H$ with $g_1 = r_1\sigma_{i_1}(h_1)$. Because of $\sigma_{i_1}(h_1)N = \sigma_{i_2}(h_1)N$, there exist $r_2 = \overline{\sigma_{i_2}(h_1)g_2} \in R_{i_2}$ and $h_2 \in H$ with $g = r_1(\sigma_{i_2}(h_1)g_2)g_3 \dots g_n N = r_1 r_2 \sigma_{i_2}(h_2) g_3 \dots g_n N$ and so on. Elements $r_i = 1$ can be omitted. In this way, g can be brought into the reduced form.

Now let $R := \{r_1 \dots r_n \sigma_{i_n}(h) : r_i \in R_i, h \in H\} \subseteq F$ be the set of reduced words. We show as in Lemma 11.6 that F acts on R. It suffices to show that each G_j acts on R. Let $x = \overline{x}\sigma_j(h_1) \in G_j$ and $r = r_1 \dots r_n \sigma_{i_n}(h) \in R$. We first assume $j \neq i_1$ and define

$$^x r := \overline{x}\,\overline{\sigma_{i_1}(h_1)r_1}\,\overline{\sigma_{i_2}(h_2)r_2} \dots \overline{\sigma_{i_n}(h_n)r_n}\sigma_{i_n}(h_{n+1}h) \in R$$

with $\sigma_{i_k}(h_k)r_k = \overline{\sigma_{i_k}(h_k)r_k}\sigma_{i_k}(h_{k+1})$ for $k = 1, \dots, n$ (in the case $x \in \sigma_j(H)$, one must remove $\overline{x}$). For $y \in G_j$ it follows that

$$^y(^x r) = \overline{yx}\,\overline{\sigma_{i_1}(h_1'h_1)r_1} \dots \overline{\sigma_{i_n}(h_n'h_n)r_n}\sigma_{i_n}(h_{n+1}'h_{n+1}h)$$

with $y\overline{x} = \overline{yx}\sigma_j(h_1')$ and $\sigma_{i_k}(h_k')\overline{\sigma_{i_k}(h_k)r_k} = \overline{\sigma_{i_k}(h_k'h_k)r_k}\sigma_{i_k}(h_{k+1}')$ for $k = 1, \dots, n$. On the other hand,

$$^{yx} r = \overline{yx}\,\overline{\sigma_{i_1}(h_1'')r_1} \dots \overline{\sigma_{i_n}(h_n'')r_n}\sigma_{i_n}(h_{n+1}''h)$$

with $yx = \overline{yx}\sigma_j(h_1'')$ and $\sigma_{i_k}(h_k'')r_k = \overline{\sigma_{i_k}(h_k'')r_k}\sigma_{i_k}(h_{k+1}'')$ for $k = 1, \dots, n$. It follows that

$$\overline{yx}\sigma_j(h_1'') = yx = y\overline{x}\sigma_j(h_1) = \overline{yx}\sigma_j(h_1'h_1)$$

and $h_1'' = h_1' h_1$, since σ_j is injective. By induction on k, we obtain

$$\begin{aligned}\overline{\sigma_{i_k}(h_k'')r_k}\sigma_{i_k}(h_{k+1}'') &= \sigma_{i_k}(h_k'')r_k = \sigma_{i_k}(h_k' h_k)r_k = \sigma_{i_k}(h_k')\overline{\sigma_{i_k}(h_k)r_k}\sigma_{i_k}(h_{k+1}) \\ &= \overline{\sigma_{i_k}(h_k' h_k)r_k}\sigma_{i_k}(h_{k+1}' h_{k+1}) = \overline{\sigma_{i_k}(h_k'')r_k}\sigma_{i_k}(h_{k+1}' h_{k+1}),\end{aligned}$$

d. h. $h_{k+1}' h_{k+1} = h_{k+1}''$. The case $j = i_1$ works analogously. Thus ${}^{y}({}^{x}r) = {}^{yx}r$ is proven and F acts on R with ${}^{x}[r] \equiv [xr] \pmod N$. For $h \in H$, $\sigma_i(h)$ has the same action on R as $\sigma_j(h)$ for $i, j \in I$. Thus N acts trivially and F/N acts on R. For $v, w \in R$ with $[v]N = [w]N$, it now holds that $v = {}^{[v]N}1 = {}^{[w]N}1 = w$. □

Lemma 11.18. *Let $G = F/N$ be an amalgam of G_I w.r.t. H and $\sigma_i \colon H \to G_i$. Then there exist subgroups $H \cong \overline{H} \leq G$ and $G_i \cong \overline{G_i} \leq G$ with $G = \langle \overline{G_i} : i \in I \rangle$ and $\overline{H} = \overline{G_i} \cap \langle \overline{G_j} : j \neq i \rangle$ for all $i \in I$.*

Proof. Let $\overline{G_i} := G_i N/N$ and $\overline{H} := \sigma_i(H)N/N \leq G$ (does not depend on i). From Lemma 11.17 it follows that $\overline{G_i} \cong G_i/G_i \cap N \cong G_i$. Because of $\overline{H} \leq \overline{G_i}$, we also have $\overline{H} \cong H$. From $F = \langle G_i : i \in I \rangle$ it follows that $G = \langle \overline{G_i} : i \in I \rangle$. Finally, $\overline{H} = \overline{G_i} \cap \langle \overline{G_j} : j \neq i \rangle$ also follows from Lemma 11.17. □

Remark 11.19. In the following, we will consider the groups G_i and H as subgroups of $G = F/N$. Let R_i be a transversal?! for G_i/H with $1 \in R_i$ for $i \in I$. Every element in G can then be uniquely written in the reduced form $r_1 \dots r_n h$ with $r_k \in R_{i_k} \setminus \{1\}$ and $h \in H$.

Lemma 11.20. *Let G be an amalgam of G_I w.r.t. H. Then:*

(i) Every element of finite order of G lies, up to conjugation, in some G_i.

(ii) If there exist distinct $i, j \in I$ with $G_i \neq H \neq G_j$, then $|G| = \infty$.

Proof.

(i) Let $g = r_1 \dots r_n h$ be in reduced form with finite order. After conjugation, we can assume $i_1 \neq i_n$ (Note: all r_i might possibly be changed). Then the elements $g, g^2, \dots$ are however all distinct. This contradiction shows $n \leq 1$ and $g \in G_{i_1}$.

(ii) For $g_1 \in G_i \setminus H$ and $g_2 \in G_j \setminus H$, $g_1 g_2$ has infinite order. □

Theorem 11.21. *Every finite group G is a subgroup of a finite group $\hat{G}$ with the following property: If $H, K \leq G$ are isomorphic subgroups and $\varphi \colon H \to K$ is an isomorphism, then there exists an $x \in \hat{G}$ with $\varphi(h) = xhx^{-1}$ for all $h \in H$. In particular, H and K are conjugate in $\hat{G}$.*

Proof. We set $\hat{G} := \mathrm{Sym}(G)$ and embed G by means of the regular representation $\sigma \colon G \to \hat{G}$, $x \mapsto \sigma_x$ with $\sigma_x(y) = xy$. Let $\hat{\varphi} \in \hat{G}$ be any bijective extension of φ. For $h \in H$ and $k \in K$ it now holds that

$$(\hat{\varphi}\sigma_h\hat{\varphi}^{-1})(k) = \varphi(h\varphi^{-1}(k)) = \varphi(h)k = \sigma_{\varphi(h)}(k)$$

and $\hat{\varphi}\sigma_h\hat{\varphi}^{-1} = \sigma_{\varphi(h)}$. □

Remark 11.22. The proof of Theorem 11.21 no longer works for infinite groups. For example, the canonical isomorphism $\varphi \colon \mathbb{Z} \to 2\mathbb{Z}$ cannot be extended to a bijection $\mathbb{Z} \to \mathbb{Z}$.

Theorem 11.23 (Higman-Neumann-Neumann)**.** *Let G be a group, $H, K \leq G$ and $\varphi\colon H \to K$ an isomorphism. Then G is a subgroup of a group $\hat{G}$ such that there exists an $x \in \hat{G}$ with $\varphi(h) = xhx^{-1}$ for all $h \in H$. In particular, H and K are conjugate in $\hat{G}$.*

Proof. Let $\langle a_i \rangle \cong \mathbb{Z}$ and $G_i := G * \langle a_i \rangle$ for $i = 1, 2$. By the universal property, there exist homomorphisms $\sigma_1\colon G * H \to G_1$ and $\sigma_2\colon G * H \to G_2$ with $(\sigma_i)_{|G} = \mathrm{id}$ and $\sigma_1(h) := a_1 h a_1^{-1}$ as well as $\sigma_2(h) := a_2 \varphi(h) a_2^{-1}$ for $h \in H$. By Lemma 11.6, σ_1 and σ_2 are injective. Let $\hat{G}$ be the amalgam of G_1 and G_2 with respect to $G * H$. By Lemma 11.18, one can identify G with $\sigma_1(G) = \sigma_2(G)$ in $\hat{G}$. For $h \in H$, it holds that $a_1 h a_1^{-1} = \sigma_1(h) = \sigma_2(h) = a_2 \varphi(h) a_2^{-1}$ and the claim follows with $x := a_2^{-1} a_1 \in \hat{G}$. □

Remark 11.24. In the situation of Theorem 11.23, $\langle G, x \rangle \leq \hat{G}$ is called an HNN *extension* of G with respect to $\varphi\colon H \to K$. If G is torsion-free, then so is every HNN extension by Lemma 11.20.

Theorem 11.25. *Every countable group is a subgroup of a group with two generators.*

Proof. Let $G = \{1 = g_0, g_1, \ldots\}$ be a countable group. If G is finite, the claim follows from $G \leq \mathrm{Sym}(G)$. Thus, let $|G| = \infty$. Let $F_2 = \langle x, y \rangle$, $X := \{y^k x y^{-k} : k \geq 0\}$ and

$$Y := \{x^k y x^{-k} g_k : k \geq 0\} \subseteq G * F_2.$$

From Lemma 11.6 it follows easily: $\langle X \rangle \cong F_X \cong F_Y \cong \langle Y \rangle$. Let $\varphi\colon \langle X \rangle \to \langle Y \rangle$, $y^k x y^{-k} \mapsto x^k y x^{-k} g_k$ for $k \geq 0$. Let $\hat{G} := \langle G * F_2, s \rangle$ be an HNN extension of $G * F_2$ with respect to φ. Then $s y^k x y^{-k} s^{-1} = x^k y x^{-k} g_k$ holds for $k \geq 0$. In particular, $y = sxs^{-1} \in \langle x, s \rangle$ and $g_k = (x^k y x^{-k})^{-1} s y^k x y^{-k} s^{-1} \in \langle x, s \rangle$ for $k \geq 1$. This shows $G \leq \hat{G} = \langle x, s \rangle$. □

Theorem 11.26. *There exists an infinite group with exactly two conjugacy classes.*

Proof. Let first $G = G_1 = \{1 = g_0, g_1, g_2, \ldots\}$ be any countable torsion-free group (for example $G = \mathbb{Z}$). Inductively, there exists a torsion-free countable HNN extension G_{n+1} of G_n such that g_1 and g_{n+1} are conjugate in G_{n+1}. In the torsion-free countable group $G^* := \bigcup_{n \in \mathbb{N}} G_n$, all g_i are then conjugate. We now set $H_1 := G$ and $H_{n+1} := H_n^*$ for $n \geq 1$. Finally, let $H := \bigcup_{n \geq 1} H_i$. For $x, y \in H \setminus \{1\}$, there exists an $n \geq 1$ with $x, y \in H_n$. By construction, x and y are conjugate in $H_{n+1} \leq H$. Thus H possesses only two conjugacy classes. □

Remark 11.27. Let G be a torsion group with class number 2. Then every non-trivial element in G has the same order p. Certainly p is a prime number. In the case $p > 2$, $x \in G \setminus \{1\}$ cannot be conjugate to x^{-1}. Thus $p = 2$ and G is abelian. It follows that $G \cong C_2$.

12 The Burnside Problem

Remark 12.1. Burnside asked in 1902:

(I) Is every finitely generated torsion group finite?

(II) Is every finitely generated periodic group finite?[30]

[30] Reminder: Periodic means that the order of all elements is globally bounded (i.e., $\exp(G) < \infty$).

(III) For $d, e \in \mathbb{N}$, are there at most finitely many finite groups with d generators and exponent e?

The first question was answered negatively by Golod in 1964. Novikov and Adjan also answered (II) negatively in 1968. Zelmanov, on the other hand, proved in 1989 that (III) is correct (he received the Fields Medal for this).[31]

Theorem 12.2 (Golod)**.** *For every prime number $p > 2$, there exists an infinite p-group with two generators.*

Proof (Gupta).
Step 1: Construction of G.
Let $\langle a \rangle \cong \langle t \rangle \cong C_p$ and $H = \langle a \rangle * \langle t \rangle$. Let $a_i := t^i a t^{-i}$ and

$$A := \langle a \rangle^H = \langle a_0, a_1, \dots, a_{p-1} \rangle.$$

Since every element of A can be uniquely written in the form $a_{i_1}^{n_1} \dots a_{i_s}^{n_s}$ with $1 \leq n_1, \dots, n_s < p$ and $i_k \neq i_{k+1}$, $A \cong \langle a_0 \rangle * \dots * \langle a_{p-1} \rangle$ (Remark 11.7). Furthermore, $H = A \rtimes \langle t \rangle$. For $0 \leq k < p$, let $\theta_k \colon A \to H$ be the homomorphism with $\theta_k(a_k) := a$ and $\theta_k(a_i) := t^{i-k}$ for $i \neq k$. Set $N_0 = 1$ and

$$N_{k+1} := \{x \in A : \forall i : \theta_i(x) \in N_k\}.$$

Clearly $1 = N_0 \leq N_1 \leq \dots$. We show $N_k \trianglelefteq H$. This is clear for $k = 0$. Let inductively $N_k \trianglelefteq H$. Then first $N_{k+1} \trianglelefteq A$. Let $w = w(a_0, \dots, a_{p-1}) \in N_{k+1}$. Because $\theta_i(a_j) = \theta_{i-1}(a_{j-1})$, we have

$$\theta_i(twt^{-1}) = \theta_i(w(a_1, \dots, a_{p-1}, a_0)) = \theta_{i-1}(w(a_0, \dots, a_{p-1})) \in N_k$$

and $twt^{-1} \in N_{k+1}$. This shows $N_{k+1} \trianglelefteq H$. Thus also $N := \bigcup_{k \geq 0} N_i \trianglelefteq H$. Finally, let

$$G := H/N.$$

Step 2: G is a finitely generated p-group.
Certainly $G = \langle aN, tN \rangle$. For $h = wt^i \in H$ with $w = a_{j_1}^{w_1} \dots a_{j_k}^{w_k} \in A$ reduced and $0 \leq i < p$, let

$$l(h) := \begin{cases} 1 + k & \text{if } i \neq 0, \\ k & \text{if } i = 0. \end{cases}$$

We show $h^{p^{l(h)}} \in N_{l(h)}$ for all $h \in H$. Then G is a p-group. For $l(h) \leq 1$, $h \in \langle t \rangle \cup \langle a_0 \rangle \cup \dots \cup \langle a_{p-1} \rangle$ and $h^p = 1 \in N_0$. Let $l(h) = n + 1$ and assume the claim is already proven for n. First assume $i \neq 0$. Then $l(w) = n$ and

$$h^p = (wt^i)^p = wt^i w t^{-i} t^{2i} w t^{-2i} \dots t^{(p-1)i} w t^{-(p-1)i}.$$

For $w = w(a_0, \dots, a_{p-1})$, we have $t^{ji} w t^{-ji} = w(a_{ji}, a_{ji+1}, \dots, a_{ji+p-1})$, where the indices are to be read modulo p. Let d_k be the number of powers of a_k in w. The number of powers of a_k in h^p is then at most

$$\sum_{j=0}^{p-1} d_{k-ji} = \sum_{j=0}^{p-1} d_j \leq n$$

[31] The proof uses Schreier's conjecture and is thus based on the CFSG.

because $p \nmid i$. The number of powers of a in $\theta_k(h^p)$ is therefore also $\leq n$. We count how many t occur in $\theta_k(h^p)$. For $r \neq k$, let s_r be the sum of the exponents of all a_r in w. The contribution of these powers to the exponent sum of t in $\theta_k(h^p)$ is then

$$\sum_{j=0}^{p-1} s_r(r+ij-k) \equiv s_r\Big(p(r-k)+i\binom{p}{2}\Big) \equiv 0 \pmod{p},$$

since $p > 2$ (note that for $ij + r = k$ nothing is counted). Because $A \trianglelefteq H$, all t in $\theta_k(h^p)$ can be shifted to the right without changing the exponent sum. This shows $\theta_k(h^p) \in A$. By shifting the t, some of the a are transformed into a_j. Their number, however, is still $\leq n$, d. h. $l(\theta_k(h^p)) \leq n$. By induction, $\theta_k(h^{p^{n+1}}) = \theta_k(h^p)^{p^n} \in N_n$ for all k, d. h. $h^{p^{n+1}} \in N_{n+1}$.

Now let $i = 0$, so $h = w \in A$. After conjugation, we can assume that w does not start and end with the same a_j. If $l(\theta_k(w)) \leq n$ already holds for all k, then $\theta_k(w^{p^n}) \in N_n$ by induction and it follows that $w^{p^n} \in N_{n+1}$ as well as $w^{p^{n+1}} \in N_{n+1}$. So let us assume $l(\theta_k(w)) = n+1$ for some k (note that the length cannot increase). If w contains two powers of a_j with $j \neq k$, then these are mapped to powers of t under θ_k. After shifting all t in $\theta_k(w)$ to the right, one obtains the contradiction $l(\theta_k(w)) \leq n$. Thus w can contain at most one power of each a_j with $j \neq k$. On the other hand, w starts and ends with different a_j. This shows $n = 2$ and after conjugation w has the form $w = a_k^r a_j^s$ with $j \neq k$. It follows that $\theta_k(w) = a^r t^{j-k}$. As in the case $i \neq 0$, it follows that $\theta_m(\theta_k(w)^p) \in A$ with $l(\theta_m(\theta_k(w)^p)) \leq 1$ for $m = 0, \ldots, p-1$ (at this point we had not yet used induction). Thus $\theta_m(\theta_k(w^{p^2})) = 1$ and $\theta_k(w^{p^2}) \in N_1$. Swapping the roles of k and j, one obtains $\theta_j(w^{p^2}) \in N_1$. For $m \notin \{k, j\}$, even $\theta_m(w^p) = 1$ holds. Thus $w^{p^2} \in N_2$ as claimed.

Step 3: $|G| = \infty$.
Assume G is finite. By Reidemeister-Schreier, N is then finitely generated. In particular, $N = N_n$ for some $n \in \mathbb{N}$. We define $v_0 := [a_1, a] \in A$ and $v_{k+1} = [v_k, a] \in A$ for $k \geq 0$. Then $\theta_0(v_0) = [t, a] = a_1 a^{-1}$ and $\theta_0(v_1) = [a_1 a^{-1}, a] = v_0$. Inductively it follows that $\theta_0(v_{k+1}) = v_k$ for $k \geq 0$. By Lemma 11.10, all v_i have infinite order, because their reduced form starts with a_1 and ends with a^{-1}. Let $r \geq 0$ with $v_0^r \in N_2$. Then $\theta_0(v_0^r) = (a_1 a^{-1})^r \in N_1$ and one obtains $(ta^{-1})^r = \theta_0(a_1 a^{-1})^r = 1$. This shows $p \mid r$. It follows that

$$1 = \big((ta^{-1})^p\big)^{r/p} = \big((at^{-1})^p\big)^{-r/p} = (a_0 a_{p-1} \ldots a_1)^{-r/p}$$

and $r = 0$. We have thus proven $\langle v_0 \rangle \cap N_2 = 1$. Now let inductively $\langle v_{k-1} \rangle \cap N_{k+1} = 1$ and $v_k^r \in N_{k+2}$. Then $v_{k-1}^r = \theta_0(v_k)^r \in N_{k+1}$ and $r = 0$. Thus $\langle v_k \rangle \cap N_{k+2} = 1$ for all $k \geq 0$. On the other hand, $v_n^{p^{l(v_n)}} \in N = N_n = N_{n+2}$ by Step 2. Contradiction. □

Definition 12.3. For $d, e \in \mathbb{N}$ let $B(d, e) := F_d / \langle g^e : g \in F_d \rangle$ be the *Burnside group* with d generators and exponent e. The Burnside problem (II) is equivalent to $|B(d, e)| < \infty$ for all $d, e \in \mathbb{N}$.

Theorem 12.4. *For $d, e \in \mathbb{N}$ it holds that*

(i) $B(1, e) \cong C_e$.

(ii) $B(d, 2) \cong C_2^d$.

(iii) $|B(d, 3)| \leq 3^{3^{d-1}}$.

Proof.

(i) Trivial.

(ii) Every group of exponent 2 is abelian. According to Burnside's basis theorem, C_2^d is the largest elementary abelian group with d generators.

(iii) Induction on d: The case $d = 1$ follows from (i). Let $d \geq 2$ and $G := B(d, 3) = \langle x_1, \ldots, x_d \rangle$. By induction, $H := \langle x_1, \ldots, x_{d-1} \rangle$ has order $\leq 3^{3^{d-2}}$. Every $g \in G$ has the form $g = h_1 x_d^{\epsilon_1} h_2 \ldots x_d^{\epsilon_n} h_{n+1} \ldots$ with $\epsilon_i \in \{\pm 1\}$ and $h_i \in H$. Because $\exp(G) = 3$, it holds that

$$x^{\pm 1} y x^{\pm 1} = y^{-1} x^{\mp 1} y^{-1} \qquad\qquad x^{\pm 1} y x^{\mp 1} = y^{-1} x^{\mp 1} y^{-1} x^{\pm 1}$$

for all $x, y \in G$. Thus $g = h_1 x_d h_2 x_d^{-1} h_3$ holds with $h_1, h_2, h_3 \in H$. It follows that $|G| \leq |H|^3 \leq 3^{3^{d-1}}$. □

Definition 12.5 (Review GT)**.** For $x_1, \ldots, x_n \in G$ let $[x_1, x_2] := x_1 x_2 x_1^{-1} x_2^{-1}$ and

$$[x_1, \ldots, x_n] := [x_1, [x_2, \ldots, x_n]]$$

for $n \geq 3$. We set $G^{[1]} := G$ and $G^{[k]} := [G, G^{[k-1]}]$ for $k \geq 2$.

Lemma 12.6. *For $G = \langle X \rangle$ and $k \geq 1$ it holds that $G^{[k]} = \langle [x_1, \ldots, x_k] : x_1, \ldots, x_k \in X \rangle G^{[k+1]}$.*

Proof. The assertion holds for $k = 1$ if one interprets $[x] = x$. Now let the assertion for some $k \geq 1$ be already proven. Let $N := \langle [x_1, \ldots, x_{k+1}] : x_1, \ldots, x_{k+1} \in X \rangle G^{[k+2]} \leq G^{[k+1]}$. For $g \in G$ and $x_1, \ldots, x_{k+1} \in X$ it holds that

$$g[x_1, \ldots, x_{k+1}] g^{-1} = [g, x_1, \ldots, x_{k+1}][x_1, \ldots, x_{k+1}] \leq G^{[k+2]} N = N.$$

From this it follows that $N \trianglelefteq G$. Modulo N, each of the generators $x \in X$ of G is permutable with the generators $[x_1, \ldots, x_k]$ and $g \in G^{[k+1]}$ of $G^{[k]}$. This shows $G^{[k+1]} = [G, G^{[k]}] \leq N$. □

Lemma 12.7 (Levi)**.** *Let G be a group of exponent 3. Then it holds that*

(i) $[x, x, y] = 1$ *for $x, y \in G$.*

(ii) $[x, y, z] = [y, z, x] = [x, z, y]^{-1}$ *for $x, y, z \in G$.*

(iii) G is nilpotent with class at most 3.

Proof.

(i) Follows from

$$x \cdot yxy^{-1} = (xy)^2 y^{-2} = (xy)^{-1} y = y^{-1} x^{-1} y = y^2 (y^{-1} x)^{-1} = y^2 (y^{-1} x)^2 = yxy^{-1} \cdot x.$$

(ii) From (i) it follows that any two conjugates of x are permutable. Thus $\langle x \rangle^G$ is abelian. It follows

$${}^x[y, z][x, z] = [xy, z] = {}^{xy}[xy, z] = {}^{xy}(\underbrace{{}^x[y, z]}_{\in \langle y \rangle^G}[x, z]) = {}^x({}^x[y, z] \cdot {}^y[x, z]) = {}^{x^{-1}}[y, z] \cdot {}^y[x, z].$$

Conjugation with x^{-1} yields

$${}^x[y, z] \cdot {}^y[x, z] = \underbrace{[y, z]}_{\in \langle z \rangle^G} \underbrace{[x, z]}_{\in \langle z \rangle^G} = [x, z][y, z].$$

Now

$$[x,y,z] = {}^{x}[y,z][z,y] = {}^{y}[z,x][x,z][y,z][z,y] = {}^{y}[z,x][x,z] = [y,z,x].$$

Because of

$$[x,y^{-1}] = [x,yy] = [x,y]\cdot {}^{y}[x,y] = [x,y]^2 = [x,y]^{-1}$$

we have $[x,y,z]^{-1} = [x,[y,z]^{-1}] = [x,z,y]$.

(iii) From (ii) it follows $[x,y,z,w] = [x,[y,z,w]] = [x,z,w,y]$ and

$$[x,y,z,w] = [x,y,[z,w]] = [x,[z,w],y]^{-1} = [y,x,[z,w]]^{-1} = [y,x,z,w]^{-1}.$$

For $\pi \in \langle (1,2),(2,3,4)\rangle = S_4$ and $x_1,\ldots,x_4 \in G$ it thus holds

$$[x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},x_{\pi(4)}] = [x_1,x_2,x_3,x_4]^{\operatorname{sgn}(\pi)}.$$

On the other hand

$$\begin{aligned}[x,y,z,w] = [x,y,[z,w]] = [[z,w],x,y] = [[z,w],[x,y]] = [[x,y],[z,w]]^{-1}\\ = [[x,y],z,w]^{-1} = [z,w,[x,y]]^{-1} \underset{\pi=(1,3)(2,4)}{=} [x,y,z,w]^{-1}.\end{aligned}$$

This shows $[x,y,z,w] = 1$ for all $x,y,z,w \in G$ and the assertion follows. □

Remark 12.8.

(i) As is well known, every group of exponent 2 is abelian, hence nilpotent of class at most 1. For p-groups with exponent $p \geq 5$, however, there is no longer an absolute bound for the nilpotency class (provided the group is nilpotent).

(ii) For $p > 10^{75}$, Olschanski proved the existence of infinite p-groups such that every non-trivial proper subgroup has order p. These groups are called *Tarski monsters* (like the sporadic monster, they are simple groups).

(iii) The next theorem improves Theorem 12.4.

Theorem 12.9 (Levi, Van der Waerden). *For $d \geq 1$ we have* $\boxed{|B(d,3)| = 3^{d(d^2+5)/6}.}$

Proof. Let $G := B(d,3) = \langle x_1,\ldots,x_d\rangle$. By von-Dyck there exists an epimorphism $G \to C_3^d$. Since G/G' is generated by d elements, it follows that $G/G' \cong C_3^d$. According to Lemma 12.6, the elementary abelian group $G'/G^{[3]}$ is generated by the elements $[x_i,x_j]G^{[3]}$ with $i<j$. This shows $|G'/G^{[3]}| \leq 3^{\binom{d}{2}}$. According to Lemma 12.7, $G^{[3]}$ is also elementary abelian with $[x,y,z] = [y,z,x]$ as well as $[x,y,z] = [x,z,y]^{-1}$. According to Lemma 12.6, $G^{[3]}$ is therefore generated by the elements $[x_i,x_j,x_k]$ with $i<j<k$. Thus $|G^{[3]}| \leq 3^{\binom{d}{3}}$ and $|G| \leq 3^k$ with $k = d + \binom{d}{2} + \binom{d}{3} = d(d^2+5)/6$.

Every element in G has the form

$$x_1^{a_1}\ldots x_d^{a_d}[x_1,x_2]^{b_{12}}\ldots[x_{d-1},x_d]^{b_{d-1,d}}[x_1,x_2,x_3]^{c_{123}}\ldots[x_{d-2},x_{d-1},x_d]^{c_{d-2,d-1,d}}$$

with $a_i,b_{ij},c_{ijk} \in \{0,1,2\}$. Let us assume that an element g possesses two different factorizations of this kind. These factorizations must be equal in G/G', because $|G/G'| = 3^d$. We can therefore assume $g \in G'$. Because of $G'' \subseteq G^{[4]} = 1$ (GT-Lemma 3.11), G' is abelian. By rearranging, one obtains a non-trivial relation of the form

$$[x_1,x_2]^{b_{12}}\ldots[x_{d-1},x_d]^{b_{d-1,d}}[x_1,x_2,x_3]^{c_{123}}\ldots[x_{d-2},x_{d-1},x_d]^{c_{d-2,d-1,d}} = 1.$$

Let $i,j,k \in \{0,1,2\}$ with $b_{ij} \neq 0$ or $c_{i,j,k} \neq 0$. By setting all other $x_l = 1$, we obtain a non-trivial relation in $B(3,3)$. If we can show $|B(3,3)| = 3^{3(3^2+5)/6} = 3^7$, we would have the desired contradiction. In GAP, $B(3,3) \cong$ `SmallGroup`$(3^7, 4487)$ can be shown.[32] Nevertheless, we provide a theoretical argument.

Let $A := \langle a,b,c,d \rangle \cong C_3^4$ and $x \in \mathrm{Aut}(A)$ with $x(a) = ad$ and $[x,b] = [x,c] = [x,d] = 1$. Obviously $x^3(a) = ad^3 = a$ and $x^3 = 1$. In $B := A \rtimes \langle x \rangle$ it holds that

$$(a^{\pm 1}x)^3 = a^{\pm 1}xa^{\pm 1}x^{-1}x^2a^{\pm 1}x^{-2} = a^{\pm 3}d^{\pm 3} = 1.$$

Thus $\exp(B) = 3$. Let $y \in \mathrm{Aut}(B)$ with $y(b) = bd^{-1}$, $y(x) = cx$ and $[y,a] = [y,c] = [y,d] = 1$. Since the images of y satisfy the same relations as a,b,c,d,x, y is indeed an automorphism. Because of $y^3(b) = bd^{-3} = b$ and $y^3(x) = c^3x = x$, y has order 3. In $C := B \rtimes \langle y \rangle$ it holds that

$$(b^ix^jy)^3 = b^ix^j \cdot b^id^{-i}c^jx^j \cdot b^id^ic^{-j}x^j = 1.$$

Thus $\exp(C) = 3$. Finally, let $z \in \mathrm{Aut}(C)$ with $y' := z(y) = a^{-1}y$, $x' := z(x) = b^{-1}x$, $c' := z(c) = cd^{-1}$ and $[z,a] = [z,b] = [z,d] = 1$. Because of

$$y'(x') = a^{-1}yb^{-1}xy^{-1}a = a^{-1}b^{-1}dyxy^{-1}a = a^{-1}b^{-1}dcxa = a^{-1}b^{-1}dcadx = c'x'$$

z is indeed an automorphism. As before, z has order 3. In $G := C \rtimes \langle z \rangle$ it holds that

$$\begin{aligned}(a^{i_a}b^{i_b}c^{i_c}d^{i_d}x^jy^kz)^3 &= a^{i_a}b^{i_b}c^{i_c}d^{i_d} \cdot a^{i_a-k}b^{i_b-j}c^{i_c+jk}d^{i_d-i_c-i_bk+i_aj} \cdot x^{2j}y^{2k}z^2a^{i_a}b^{i_b}c^{i_c}d^{i_d}x^jy^kz \\ &= a^{i_a}b^{i_b}c^{i_c}d^{i_d} \cdot a^{i_a-k}b^{i_b-j}c^{i_c+jk}d^{i_d-i_c-i_bk+i_aj} \cdot a^{i_a+k}b^{i_b+j}c^{i_c-jk}d^{i_d+i_c+i_bk-i_aj} = 1,\end{aligned}$$

i.e., $\exp(G) = 3$. Since $G = \langle x,y,z \rangle$ and $|G| = 3^7$, it follows that $G \cong B(3,3)$. □

Theorem 12.10 (Sanov)**.** *For $d \geq 1$, it holds that $|B(d,4)| < \infty$.*

Proof. Let $G := B(d,4)$. Wlog. let $d \geq 2$. By induction, $H := \langle x_1, \ldots, x_{d-1} \rangle$ is finite. Let $y := x_d^2$. We first show that $K := \langle y, H \rangle$ is finite. The claim then follows by the same argument for K and x_d instead of H and y. Every $g \in K$ has the form $g = h_1yh_2 \ldots yh_n$ with $h_1, \ldots, h_n \in H$. Let n be as small as possible. Assume $n \geq 2|H| + 3$. Then among the elements $h_2h_3^{-1}$, $h_2h_4(h_3h_5)^{-1}$, $h_2h_4(h_3h_5h_7)^{-1}, \ldots$ there are two that are equal, say

$$h_2h_4 \ldots h_{2r}(h_3h_5 \ldots h_{2r+1})^{-1} = h_2 \ldots h_{2s}(h_3 \ldots h_{2s+1})^{-1}$$

with $r < s \leq (n-1)/2$. Thus

$$h_{2r+2}h_{2r+4} \ldots h_{2s}(h_{2r+3}h_{2r+5} \ldots h_{2s+1})^{-1} = 1.$$

Because $\exp(G) = 4$ and $y^2 = 1$, it holds that

$$yhy = h^{-1}y^{-1}h^{-1}y^{-1}h^{-1} = h^{-1}yh^{-1}yh^{-1}$$

for $h \in H$. We apply this rule to $h = h_{2s+1}$ in g:

$$g = h_1yh_2 \ldots h_{2s}h_{2s+1}^{-1}yh'_{2s+1}y \ldots yh_n.$$

By applying it again with $h = h_{2s}h_{2s+1}^{-1}$, h_{2s-1} is replaced by $h_{2s-1}(h_{2s}h_{2s+1}^{-1})^{-1} = h_{2s-1}h_{2s+1}h_{2s}^{-1}$. After that, h_{2s-2} is replaced by $h_{2s-2}h_{2s}(h_{2s-1}h_{2s+1})^{-1}$ etc. Finally, h_{2r+2} is replaced by 1. This contradicts the minimality of n. Consequently, $n \leq 2|H| + 2$ and one obtains $|K| < \infty$. □

[32] For example with `OneGroup`$(3^7,$ `RankPGroup`$, 3,$ `Exponent`$, 3)$;

Example 12.11. Let $F := F_2$ and $N := \langle x^2 : x \in F \rangle$. Then $|F/N| = 4$ and Schreier's formula shows $N \cong F_5$. Therefore $M := \langle x^2 : x \in N \rangle$ has index 32 in N. Since N is characteristic in G, it holds that $M \trianglelefteq G$. Now $G = F/M$ has order 2^7, exponent 4 and two generators. This shows $|B(2,4)| \geq 2^7$. The same argument yields $|B(2,8)| \geq 2^{7+2^7+1} = 2^{136}$ and $|B(3,4)| \geq 2^{3+17} = 2^{20}$.

Remark 12.12. It holds that $|B(2,4)| = 2^{12}$, $|B(3,4)| = 2^{69}$, $|B(4,4)| = 2^{422}$ and $|B(5,4)| = 2^{2728}$. Hall proved

$$\boxed{|B(d,6)| = 2^{1+3^a(d-1)}3^{b+\binom{b}{2}+\binom{b}{3}}}$$

with $a = d + \binom{d}{2} + \binom{d}{3}$ and $b = 1 + 2^d(d-1)$. It is known that $|B(d,e)| = \infty$ for all $d \geq 2$ and $e \geq 8000$ (and all odd $e \geq 557$).[33][34]

Definition 12.13.

- Let M be the intersection of all normal subgroups of $B(d,e)$ with finite index and $B_0(d,e) := B(d,e)/M$. The Burnside Problem (III) is equivalent to $|B_0(d,e)| < \infty$ for all $d, e \in \mathbb{N}$.
- Let G be a finite group and p a prime. We define $N_0 := 1$ and

  $$N_{2k-1}/N_{2k-2} := \mathrm{O}_{p'}(G/N_{2k-2}), \qquad N_{2k}/N_{2k-1} := \mathrm{O}_p(G/N_{2k-1})$$

  for $k \geq 1$. If there exists a $k \geq 0$ with $N_k = G$, then G is called *p-solvable*. If applicable, the smallest number $l := l_p(G) \geq 0$ with $N_{2l+1} = G$ is called the *p-length* of G.

Remark 12.14.

(i) If G is p-solvable, then the normal series $N_0 < \ldots < N_k$ can be refined to a chief series and a composition series. Therefore, all chief and composition factors are p-groups or p'-groups. In particular, every minimal normal subgroup of G is a p-group or a p'-group.

(ii) Obviously, every solvable group is p-solvable for every prime p.

Lemma 12.15 (Hall-Higman)**.** *Let G be a p-solvable group with $\mathrm{O}_{p'}(G) = 1$. Then*

$$\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G).$$

Proof. Wlog. $N := \mathrm{O}_p(G)$. Then $\mathrm{C}_G(N)N/N \trianglelefteq G/N$. In the case $\mathrm{C}_G(N) \not\subseteq N$, there exists a minimal normal subgroup $M/N \trianglelefteq G/N$ with $M \leq \mathrm{C}_G(N)N$. Because of $\mathrm{O}_p(G/N) = 1$, M/N is a p'-group (Remark 12.14). By Schur-Zassenhaus, $M = N \rtimes H$. Since $\mathrm{C}_G(N)N/\mathrm{C}_G(N) \cong N/\mathrm{Z}(N)$ is a p-group, it follows that $H \leq \mathrm{C}_G(N)$ and $M = N \times H$. But then $H \leq \mathrm{O}_{p'}(M) \leq \mathrm{O}_{p'}(G) = 1$. □

Theorem 12.16. *Let G be p-solvable and c_p the nilpotency class of a Sylow p-subgroup of G. Then $l_p(G) \leq c$.*

Proof. Induction on c: In the case $c = 0$, $G = \mathrm{O}_{p'}(G)$ and $l_p(G) = 0$. Now let $c > 0$ and $P \in \mathrm{Syl}_p(G)$. Wlog. let $\mathrm{O}_{p'}(G) = 1$ and $N := \mathrm{O}_p(G) > 1$. By Hall-Higman, $\mathrm{Z}(P) \leq \mathrm{C}_G(P) \leq \mathrm{C}_G(N) \leq N$. The Sylow p-subgroup

$$PN/N \cong P/P \cap N \cong (P/\mathrm{Z}(P))/((N \cap P)/\mathrm{Z}(P))$$

of G/N thus has nilpotency class $\leq c - 1$. By induction, $l_p(G) = l_p(G/N) + 1 \leq c$. □

[33] See `https://arxiv.org/abs/2303.15997v5`

[34] Magnus commented on the first work of this kind as "This paper is possibly the most difficult paper to read that has ever been written on mathematics."

Theorem 12.17. *For $d \geq 1$, $|B_0(d,6)| \leq |B_0(d,12)| < \infty$.*

Proof. Since $B_0(d,6)$ is a factor group of $B_0(d,12)$, it suffices to show $|B_0(d,12)| < \infty$. Let G be a finite group with d generators and exponent 12. According to Burnside's p^aq^b-theorem, G is solvable. A 3-Sylow subgroup P of G has exponent ≤ 3 and nilpotency class ≤ 3 according to Lemma 12.7. According to Theorem 12.16, $l_3(G) \leq 3$. So let $1 = N_0 \leq \ldots \leq N_7 = G$ be as in Definition 12.13. Then G/N_6 is a 2-group with d generators and exponent ≤ 4. According to Theorem 12.10, $|G/N_6|$ is bounded by a function in d. According to Reidemeister-Schreier, the number of generators of N_6 is bounded by a function in d. Furthermore, N_6/N_5 has exponent ≤ 3. According to Theorem 12.9, $|N_6/N_5|$ is also bounded by a function in d. Continuing in this manner, one obtains an upper bound for $|G|$. □

Remark 12.18. Hall and Higman showed that one can estimate the p-length of G by a function in the exponent of a p-Sylow subgroup of G. With the help of the CFSG, the question $|B_0(d,n)| < \infty$ can now be reduced to the case where $n = p^k$ is a prime power. Zelmanov finally proved $|B_0(d,p^k)| < \infty$.

Definition 12.19. Let $G = \langle x_1, \ldots, x_n \rangle$. We recursively define a set of *basic commutators* $c_1, c_2, \ldots$ as follows:

- For $i = 1, \ldots, n$, let $c_i := x_i$ be the basic commutators with *weight* $\omega(c_i) := 1$.
- Let the basic commutators $c_1, \ldots, c_k$ be already defined. For $1 \leq i < j \leq k$, $c := [c_i, c_j]$ is a basic commutator if either $\omega(c_j) = 1$ or $c_j = [c_s, c_t]$ with $i \geq s$ holds. If applicable, let $\omega(c) := \omega(c_i) + \omega(c_j)$.
- The basic commutators are numbered increasingly by weight, where the order of the commutators with the same weight does not matter.

For $k \in \mathbb{N}$, let $\delta_n(k)$ be the (finite) number of basic commutators with weight k. Here, basic commutators are considered distinct even if they represent the same group elements (e. g. if G is abelian). We show in Remark 12.31 that in the free group $G = F_n$, basic commutators are indeed pairwise distinct.

Example 12.20. Obviously, $\delta_n(1) = n$ and $\delta_n(2) = \binom{n}{2}$ hold. For $n = 3$ and $(x_1, x_2, x_3) = (x, y, z)$,

$$x, y, z, [x,y], [x,z], [y,z], [x,x,y], [x,x,z], [y,x,y], [y,x,z], [y,y,z], [z,x,y], [z,x,z], [z,y,z]$$

are the basic commutators with weight ≤ 3. Thus, $\delta_3(3) = 8$ holds.

Theorem 12.21. *Let $G = \langle x_1, \ldots, x_n \rangle$ and $k \in \mathbb{N}$. Then $G^{[k]}/G^{[k+1]}$ is generated by the cosets of the basic commutators with weight k.*

Proof. For $k = 1$, the claim is clear because $G^{[1]} = G$. Let the statement already be proven for $k-1$. By definition, $G^{[k]}$ is generated by the commutators $[g,h]$ with $g \in G$ and $h \in G^{[k-1]}$. Because of $G^{[k]}/G^{[k+1]} \leq \mathrm{Z}(G/G^{[k+1]})$, it holds that

$$\begin{aligned} [xy,h] &= {}^x[y,h][x,h] \equiv [x,h][y,h] \pmod{G^{[k+1]}}, \\ [x^{-1},h] &= [x^{-1},h][xx^{-1},h]^{-1} \equiv [x^{-1},h]\big([x,h][x^{-1},j]\big)^{-1} \equiv [x,h]^{-1} \pmod{G^{[k+1]}} \end{aligned}$$

for all $x, y \in G$. We can therefore assume $g = x_i = c_i$ with $1 \leq i \leq n$. By induction, there exist basic commutators $c_{j_1}, \ldots, c_{j_l}$ with weight $k-1$ and $h' \in G^{[k]}$ with $h = c_{j_1} \ldots c_{j_l} h'$. It now holds analogously

$$[g,h] \equiv [g,c_{j_1}] \ldots [g,c_{j_l}][g,h'] \equiv [g,c_{j_1}] \ldots [g,c_{j_l}] \pmod{G^{[k+1]}}.$$

We can thus assume $h = c_j$. In the case $k = 2$, either $[c_i, c_j]$ or $[c_j, c_i] = [c_i, c_j]^{-1}$ is a basic commutator. So let $k \geq 3$ and $c_j = [c_s, c_t]$ with $s < t$. In the case $i \geq s$, $[c_i, c_j]$ is again a basic commutator. Therefore, let $i < s < t$. The Hall-Witt identity (GT-Lemma 3.6) simplifies to

$$1 = {}^{c_s}[c_i, c_s^{-1}, c_t] \cdot {}^{c_t}[c_s, c_t^{-1}, c_i] \cdot {}^{c_i}[c_t, c_i^{-1}, c_s] \equiv [c_i, c_s^{-1}, c_t][c_s, c_t^{-1}, c_i][c_t, c_i^{-1}, c_s] \pmod{G^{[k+1]}}.$$

There exists a $z \in G^{[k]}$ with $[c_s^{-1}, c_t] = [c_s, c_t]^{-1} z$ and

$$[c_i, c_s^{-1}, c_t] \equiv [c_i, [c_s, c_t]^{-1}] \equiv [c_i, c_j]^{-1} \pmod{G^{[k+1]}}.$$

Because of $[G^{[a]}, G^{[b]}] \leq G^{[a+b]}$ (GT-Lemma 3.11), one analogously obtains

$$[c_s, c_t^{-1}, c_i] \equiv [c_s, [c_t, c_i]^{-1}] \equiv [c_s, c_i, c_t] \pmod{G^{[k+1]}}$$

and $[c_t, c_i^{-1}, c_s] \equiv [c_t, c_i, c_s]^{-1} \pmod{G^{[k+1]}}$. Overall,

$$[c_i, c_j] \equiv [c_s, c_i, c_t][c_t, c_i, c_s]^{-1} \pmod{G^{[k+1]}}.$$

Let $c_u := [c_i, c_t]$ and $c_v := [c_i, c_s]$. Because of $\omega(c_u) = \omega(c_t) + 1$, it holds that $s < t < u$. Therefore, $[c_s, c_i, c_t] = [c_s, c_u]$ is a basic commutator. In the case $t < v$, $[c_t, c_i, c_s] = [c_t, c_v]$ is also a basic commutator. In the case $t = v$, $[c_t, c_v] = 1$. Thus, the case $t > v$ remains. Let $c_t = [c_w, c_y]$. Since $c_j = [c_s, c_t]$ is a basic commutator, $s \geq w$ holds. Because of $\omega(c_v) = \omega(c_s) + 1$, it holds that $v > s \geq w$. This shows that $[c_t, c_v]^{-1} = [c_v, c_t]$ is a basic commutator. □

Corollay 12.22. *Every nilpotent group that is generated by finitely many elements of finite order is finite.*

Proof. Let $G = \langle x_1, \ldots, x_n \rangle$ be nilpotent with elements $x_1, \ldots, x_n$ of finite order. Then $|G/G'| < \infty$. Inductively, we can assume $e := |G/G^{[k]}| < \infty$. Wlog. let $G^{[k+1]} = 1$. According to Theorem 12.21, $G^{[k]}$ is generated by finitely many commutators. From GT-Exercise 18, it follows that

$$[x, y]^e = [x^e, y] \in G^{[k+1]} = 1$$

for all $[x, y] \in G^{[k]} \leq \mathrm{Z}(G)$. Thus, $G^{[k]}$ is also finite. □

Remark 12.23. The proof shows: If a nilpotent group G is generated by finitely many p-elements, then G is a p-group.

Lemma 12.24. *Let $c_1, c_2, \ldots$ be the basic commutators of $G = \langle x_1, \ldots, x_n \rangle$. For all $k \geq 1$ there are exactly n^k sequences $(a_1, a_2, \ldots) \in \mathbb{N}_0^\infty$ with $a_1 \geq a_2 \geq \ldots$ and $\sum_{a_i \geq 1} \omega(c_{a_i}) = k$.*

Proof. For $s \in \mathbb{N}$ let F_s be the set of all sequences $(a_1, a_2, \ldots) \in \mathbb{N}_0^\infty$, such that:

(i) There exists an $l \in \mathbb{N}_0$ with $a_1, \ldots, a_{l-1} \geq s$, $a_l > s$, $s \geq a_{l+1} \geq a_{l+2} \geq \ldots$.

(ii) If $c_{a_m} = [c_i, c_j]$ for an $m \leq l$, then $i < s$.

(iii) $\sum_{a_i \geq 1} \omega(c_{a_i}) = k$.

If s is large enough, then $\omega(c_s) > k$. If necessary, $l = 0$ in (i) and (ii) provides no restrictions. Therefore, F_s consists exactly of the sequences we want to count. For $s = 1$ it holds that $\omega(c_{a_i}) = 1$ for all $i \in \mathbb{N}$. According to (iii), then $a_k > 0 = a_{k+1} = a_{k+2} = \ldots$. Otherwise, there are no further restrictions on the sequence. Because of $a_i \in \{1, \ldots, n\}$ it follows that $|F_1| = n^k$.

For a given $s \geq 1$ it suffices to construct a bijection $\varphi\colon F_s \to F_{s+1}$. Let $a := (a_i) \in F_s$ and l as in (i). We traverse the sequence from right to left starting at a_l. If we encounter a pair $(a_i, a_{i+1}) = (s, t)$ with $t > s$, then $c_j := [c_s, c_t]$ is a basic commutator according to (ii). We replace (a_i, a_{i+1}) by j if necessary. This does not change (iii). Subsequently, we consider the pair (a_{i-1}, j) and iterate. The sequence constructed in this way is denoted by $\varphi(a)$. By construction, $\varphi(a)_i \geq s + 1$ for $i = 1, \ldots, l$. In the case $\varphi(a)_1 = \ldots = \varphi(a)_l = s + 1$, (i) holds with $l = 0$ with respect to $s + 1$. Otherwise, (i) holds with $l := \max\{i : a_i > s + 1\}$. By construction, $\varphi(a)$ also satisfies (ii). Therefore, $\varphi(a) \in F_{s+1}$.

Conversely, let $(b_i)_i \in F_{s+1}$ with l as in (i) be given. We traverse b from left to right starting at b_1. If $c_{b_i} = [c_s, c_t]$ for an $i \leq l$, we replace b_i by the pair (s, t). Subsequently, we consider t or c_t respectively. This process must end after finitely many steps because of $\sum_{b_j \geq 1} \omega(c_{b_j}) = k$. The final result is denoted by $\psi(b)$. Obviously, $\psi(b) \in F_s$ and the maps φ and ψ are inverse bijections to each other. □

Lemma 12.25. *For $k \in \mathbb{N}$ it holds that*

$$\sum_{d|k} \delta_n(d) d = n^k.$$

Proof. Let S_k be the set of sequences $(a_1, a_2, \ldots) \in \mathbb{N}_0^\infty$ from Lemma 12.24. If a basic commutator c_j occurs exactly t times in $(a_i)_i \in S_k$, then it contributes $t\omega(c_j)$ to the sum $\sum_{a_i \geq 1} \omega(c_{a_i}) = k$. Let K_w be the set of basic commutators with weight w. From Lemma 12.24 one obtains an identity of formal power series:

$$\begin{aligned}\prod_{w=1}^{\infty} (1 - X^w)^{-\delta_n(w)} &= \prod_{w=1}^{\infty} (1 + X^w + X^{2w} + \ldots)^{\delta_n(w)} = \prod_{w=1}^{\infty} \prod_{c \in K_w} (1 + X^{\omega(c)} + X^{2\omega(c)} + \ldots) \\ &= \prod_{i=1}^{\infty} (1 + X^{\omega(c_i)} + X^{2\omega(c_i)} + \ldots) = 1 + \sum_{k=1}^{\infty} \sum_{(a_i) \in S_k} X^k = \sum_{k=0}^{\infty} n^k X^k = (1 - nX)^{-1}.\end{aligned}$$

We apply the formal logarithm $\log(1 - X) := -\sum_{k=1}^{\infty} \frac{1}{k} X^k$ to both sides:[35]

$$\sum_{k=1}^{\infty} \frac{n^k}{k} X^k = -\log(1 - nX) = -\sum_{w=1}^{\infty} \delta_n(w) \log(1 - X^w) = \sum_{w=1}^{\infty} \sum_{k=1}^{\infty} \frac{\delta_n(w)}{k} X^{wk} = \sum_{k=1}^{\infty} \sum_{d|k} \frac{d\delta_n(d)}{k} X^k.$$

A comparison of coefficients shows the claim. □

Theorem 12.26 (Witt's formula)**.** *Let $G = \langle x_1, \ldots, x_n \rangle$ and $k \in \mathbb{N}$. Then $G^{[k]}/G^{[k+1]}$ is generated by*

$$\boxed{\delta_n(k) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}}$$

elements. Here μ is the Möbius function from number theory.

[35] See Combinatorics notes

Proof. According to Theorem 12.21, $G^{[k]}/G^{[k+1]}$ is generated by the $\delta_n(d)$ basic commutators with weight k. The formula for $\delta_n(d)$ follows by Möbius inversion from Lemma 12.25. □

Remark 12.27. We show next that the estimate for the number of generators of $G^{[k]}/G^{[k+1]}$ in Theorem 12.26 is optimal for the free group.

Definition 12.28.

- Let $M := \langle X_1, \ldots, X_n \rangle$ be the monoid of all monomials $X_{i_1} \ldots X_{i_k}$ with $k \geq 0$ and $1 \leq i_1, \ldots, i_k \leq n$ in the non-commuting variables $X_1, \ldots, X_n$ with respect to concatenation (i.e., the free group without inverses).
- Let R be a commutative ring and $A := RM$ the free R-module with basis M. By extending the concatenation distributively, A becomes a ring. For example,

  $$(1 + X_2X_1^2)(X_1 + X_3X_2) = X_1 + X_2X_1^3 + X_3X_2 + X_2X_1^2X_3X_2.$$

  One calls A the *free algebra* over $X_1, \ldots, X_n$.
- Let $\deg(X_{i_1} \ldots X_{i_k}) := k$ be the *degree* of a monomial. Let A_k be the free R-submodule of A generated by the monomials of degree k, i.e., A_k consists of all *homogeneous polynomials* of degree k. Obviously, $\mathrm{rk}(A_k) = n^k$ and $A = \bigoplus_{k \geq 0} A_k$.
- The *commutator* of $a, b \in A$ is $[a, b] := ab - ba$. The basic commutators in A are defined analogously to Definition 12.19. Instead of c_i we write C_i. For $n = 2$ and $(X_1, X_2) = (X, Y)$, for example,

  $$C_4 = [C_1, C_3] = X[X, Y] - [X, Y]X = X(XY - YX) - (XY - YX)X = X^2Y - 2XYX + YX^2$$

  (cf. Example 12.20).

Lemma 12.29. *For all $k \in \mathbb{N}$,*

$$B_k := \left\{ C_{a_1} \ldots C_{a_m} : m \in \mathbb{N},\ a_1 \geq \ldots \geq a_m,\ \sum_{i=1}^{m} \omega(c_{a_i}) = k \right\}$$

is a basis of A_k. In particular, the basic commutators with weight k are linearly independent over $\mathbb{Z}$.

Proof. One easily sees inductively that a basic commutator with weight k lies in A_k. Therefore $B_k \subseteq A_k$ holds. According to Lemma 12.24, $|B_k| = n^k = \mathrm{rk}(A_k)$. We first show that B_k is a generating system of A_k. For this, we consider the sets F_s from the proof of Lemma 12.24. If s is large enough, then B_k consists exactly of the products $C_{a_1} \ldots C_{a_m}$ with $a = (a_i)_i \in F_s$ (zeros are removed from the sequence). On the other hand, F_1 corresponds exactly to the monomials $X_{i_1} \ldots X_{i_k}$. It is therefore sufficient to show that one can express $C := C_{a_1} \ldots C_{a_m}$ with $a \in F_s$ by a linear combination of $C_{b_1} \ldots C_{b_{m'}}$ with $b \in F_{s+1}$. As in the proof of Lemma 12.24, we traverse the sequence a from right to left. If $(a_i, a_{i+1}) = (s, t)$ with $t > s$, we replace C_sC_t in C by the equivalent expression $[C_s, C_t] + C_tC_s$. With $C_j = [C_s, C_t]$, it holds that

$$C = C_{a_1} \ldots C_{a_{i-1}} C_j C_{a_{i+2}} \ldots C_{a_m} + C_{a_1} \ldots C_{a_{i+1}} C_{a_i} \ldots C_{a_m}.$$

With the first summand, we proceed as before. In the second summand, $C_{a_i} = C_s$ has moved to the right. The corresponding sequence still lies in F_s and can be treated like the initial sequence. After finitely many steps, one reaches a linear combination of $C_{b_1} \ldots C_{b_{m'}}$ with $b \in F_{s+1}$.

Thus B_k is a generating system of A_k.[36] There is therefore a matrix $M \in R^{n^k \times n^k}$ such that the map $\varphi\colon R^{n^k} \to A_k \cong R^{n^k}$, $v \mapsto Mv$ is surjective. For the i-th standard basis vector $e_i \in R^{n^k}$, there exists an $x_i \in R^{n^k}$ with $Mx_i = e_i$. Let $L \in R^{n^k \times n^k}$ be the matrix with columns $x_1, \ldots, x_{n^k}$. Then $ML = 1_{n^k}$ and $\det(M) \in R^\times$ holds. For the matrix M^* complementary to M, one obtains $\det(M)^{-1}M^*M = 1_{n^k}$. This shows that φ is injective. Thus B_k is linearly independent. □

Definition 12.30. As before, let $A := R\langle X_1, \ldots, X_n\rangle$. Let $I := A_{>k} := \sum_{l>k}^{\infty} A_l$ be the ideal in A generated by A_{k+1}. Let $\overline{A} := A/I$ and $\mathcal{A} := (1 + A_{>0} + I)/I \subseteq \overline{A}$.

Remark 12.31. For $a = 1 + a_1 + I \in \mathcal{A}$, it holds that

$$a(1 - a_1 + a_1^2 \mp \ldots \pm a_1^k + I) = 1 \pm a_1^{k+1} + I = 1.$$

Therefore $\mathcal{A} \leq \overline{A}^\times$.

Theorem 12.32. *Let F be the free group of rank n and $k \in \mathbb{N}$. Then $F^{[k]}/F^{[k+1]}$ is a free abelian group of rank $\delta_n(k)$.*

Proof. Let F be free w.r.t. $x_1, \ldots, x_n$. We consider $A = \mathbb{Z}\langle X_1, \ldots, X_n\rangle$ with $R = \mathbb{Z}$. By the universal property of F, there exists a homomorphism $\varphi\colon F \to \mathcal{A}$ with $\varphi(x_i) = 1 + X_i + I$. We show

$$\varphi(c_i) \equiv 1 + C_i \pmod{\overline{A_{>\omega(c_i)}}}$$

for every basic commutator c_i. This is clear for $\omega(c_i) = 1$. Let $w_i := \omega(c_i)$ and $w_j := \omega(c_j)$. By induction, there exist $\epsilon_i, \epsilon_i' \in \overline{A_{>w_i}}$ and $\epsilon_j, \epsilon_j' \in \overline{A_{>w_j}}$ with $\varphi(c_i) = 1 + C_i + \epsilon_i$, $\varphi(c_i)^{-1} = 1 - C_i + \epsilon_i'$, $\varphi(c_j) = 1 + C_j + \epsilon_j$ and $\varphi(c_j)^{-1} = 1 - C_j + \epsilon_j'$ (Remark 12.31). From

$$1 = \varphi(c_i)\varphi(c_i)^{-1} = (1 + C_i + \epsilon_i)(1 - C_i + \epsilon_i') = 1 + \epsilon_i + \epsilon_i' - C_i^2 + C_i\epsilon_i' - \epsilon_i C_i + \epsilon_i\epsilon_i'$$

it follows that

$$\begin{aligned}
\varphi([c_i, c_j]) &= (1 + C_i + \epsilon_i)(1 + C_j + \epsilon_j)(1 - C_i + \epsilon_i')(1 - C_j + \epsilon_j') \\
&\equiv 1 + C_i + C_j - C_i - C_j + \epsilon_i + \epsilon_j + \epsilon_i' + \epsilon_j' + C_iC_j - C_iC_j - C_jC_i + C_iC_j \\
&\quad - C_i^2 + C_i\epsilon_i' - \epsilon_i C_i + \epsilon_i\epsilon_i' - C_j^2 + C_j\epsilon_j' - \epsilon_j C_j + \epsilon_j\epsilon_j' \pmod{\overline{A_{>w_i+w_j}}} \\
&\equiv 1 + [C_i, C_j] \pmod{\overline{A_{>\omega([c_i,c_j])}}},
\end{aligned}$$

as claimed. This implies $\varphi(F^{[k+1]}) = 1$. Let $c_{i_1}, \ldots, c_{i_m}$ be the basic commutators of F with weight k. By Theorem 12.21, $F^{[k]}/F^{[k+1]}$ is generated by $c_{i_1}F^{[k+1]}, \ldots, c_{i_m}F^{[k+1]}$. Let $a_1, \ldots, a_m \in \mathbb{Z}$ with $g := c_{i_1}^{a_1} \ldots c_{i_m}^{a_m} \in F^{[k+1]}$. Then

$$1 = \varphi(g) = (1 + C_{i_1})^{a_1} \ldots (1 + C_{i_m})^{a_m} = 1 + a_1C_{i_1} + \ldots + a_mC_{i_m}.$$

From Lemma 12.29 it follows that $a_1 = \ldots = a_m = 0$. Thus $c_{i_1}F^{[k+1]}, \ldots, c_{i_m}F^{[k+1]}$ is a basis of $F^{[k]}/F^{[k+1]}$. □

Remark 12.33. By Remark 2.10, F_2' is free with (countably) infinite rank. Therefore F_2'/F_2'' is not finitely generated.

[36] Since R is in general not a field, linear independence does not follow automatically.

Corollay 12.34. *The free nilpotent group $F_n/F_n^{[k+1]}$ of rank n and nilpotency class k is torsion-free.*

Theorem 12.35. *For every prime power $q = p^e$, the following holds:*

(i) There exists exactly one group P with n generators, exponent q, nilpotency class $p-1$ and $P^{[k]}/P^{[k+1]} \cong C_q^{\delta_n(k)}$ for $k = 1, \ldots, p-1$.

(ii) There exists a group P with n generators, exponent q and nilpotency class $q-1$, such that $P^{[k]}/P^{[k+1]}$ has rank $\delta_n(k)$ for $k = 1, \ldots, q-1$.

Proof. Let F be the free group with generators $x_1, \ldots, x_n$. Let $N := \langle g^q : g \in F \rangle \trianglelefteq F$ and $\overline{F} := F/N$.

(i) We consider $A := R\langle X_1, \ldots, X_n \rangle$ with $R = \mathbb{Z}/q\mathbb{Z}$. In the definition of $\mathcal{A}$, let $k < p$. For $1 + a \in \mathcal{A}$, it then holds that

$$(1+a)^q = 1 + qa + \binom{q}{2}a^2 + \ldots + \binom{q}{p}a^p + \ldots = 1.$$

For the mapping $\varphi\colon F \to \mathcal{A}$, $x_i \mapsto 1 + X_i + I$ from the proof of Theorem 12.32, it thus holds that $\varphi(N) = 1$. The cosets of the basic commutators $c_{i_1}, \ldots, c_{i_m} \in F$ with weight k form, as before, a generating system of $\overline{F}^{[k]}/\overline{F}^{[k+1]}$. Let $a_1, \ldots, a_m \in R$ with $g := c_{i_1}^{a_1} \ldots c_{i_m}^{a_m} \in NF^{[k+1]}$. Then, as before, $\varphi(g) = 1$ and $a_1 = \ldots = a_m = 0$. This shows that $\overline{F}^{[k]}/\overline{F}^{[k+1]} \cong F^{[k]}N/F^{[k+1]}N$ is a free R-module with rank $\delta_n(k)$. Thus $P := \overline{F}/\overline{F}^{[p]} \cong F/NF^{[p]}$ has the desired properties.

Conversely, let Q be a p-group with the specified properties. Then there exists an epimorphism $\sigma\colon F \to Q$ with $NF^{[p]} \leq \mathrm{Ker}(\sigma)$. In particular, $|P| \geq |Q|$. From

$$|Q| = q^{\delta_n(1)+\ldots+\delta_n(p-1)} = |P|$$

it follows that $Q \cong P$.

(ii) This time we consider $A = \mathbb{F}_p\langle X_1, \ldots, X_n \rangle$ and $k < q$ in the definition of $\mathcal{A}$. For $1 + a \in \mathcal{A}$, it still holds that

$$(1+a)^q = \sum_{k=0}^{q} \binom{q}{k} a^k = 1$$

because of $\binom{q}{k} \equiv 0 \pmod{p}$ for $k = 1, \ldots, q-1$. One thus again obtains a homomorphism $\varphi\colon F \to \mathcal{A}$ with $\varphi(N) = 1$. Since the basic commutators with weight k are linearly independent over $\mathbb{F}_p$, $\overline{F}^{[k]}/\overline{F}^{[k+1]}$ has rank $\delta_n(k)$ (but not necessarily exponent q). To show that $P := \overline{F}/\overline{F}^{[q]} \cong F/NF^{[q]}$ has exponent q, we consider an epimorphism $\psi\colon F \to C_q^n$. Because of $\psi(NF') = 1$, we have $P/P' \cong F/F'N \cong C_q^n$. □

Remark 12.36.

(i) Theorem 12.35 shows $\lim_{p\to\infty} \log_p |B(n,p)| = \infty$ for $n \geq 2$. For the group P from Theorem 12.35(i), more precisely, it holds that

$$\log_q |P| = \sum_{k=1}^{q-1} \delta_n(k) = \sum_{k=1}^{q-1} \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d} = \sum_{d=1}^{q-1} \frac{\mu(d)}{d} \sum_{e=1}^{\lfloor (q-1)/d \rfloor} \frac{n^e}{e}.$$

(ii) As is well known, every group with exponent 2 is abelian, i. e. the nilpotency class is ≤ 1. Thus, Theorem 12.35 cannot be transferred to higher nilpotency classes.

(iii) Let $p = q > 2$ and P be the group from Theorem 12.35(i). Then $P/P^{[3]}$ is a maximal Schur extension of C_p^n (Example 5.21).

(iv) In Theorem 12.35(ii), $P^{[k]}/P^{[k+1]}$ does not necessarily have exponent q. In fact, there is no group with two generators, order 2^6, exponent 4 and nilpotency class 2:

```
OneGroup(2^6,RankPGroup,2,Exponent,4,NilpotencyClassOfGroup,2);
```

Example 12.37. For $q \in \{3,5\}$ one obtains $|B(n,3)| \geq 3^{n+\binom{n}{2}} = 3^{\binom{n+1}{2}}$ (cf. Theorem 12.9) and $|B(2,5)| \geq 5^{2+1+2+3} = 5^8$. In fact, $|B_0(2,5)| = 5^{34}$ holds (without proof).

Remark 12.38. Uniquely determined groups with certain properties as in Theorem 12.35 will be constructed more generally in Theorem 13.29.

13 Group Classes and Varieties

Definition 13.1. A class $\mathcal{X}$ of groups is called a *group class*, if $1 \in \mathcal{X}$ and $G \cong H \in \mathcal{X} \Rightarrow G \in \mathcal{X}$. The elements of $\mathcal{X}$ are called $\mathcal{X}$-groups or groups with property $\mathcal{X}$. A group G is called a *residual $\mathcal{X}$-group*, if for all $g \in G \setminus \{1\}$ there exists an $N \trianglelefteq G$ with $g \notin N$ and $G/N \in \mathcal{X}$. The residual $\mathcal{X}$-groups form the group class $\mathcal{X}^r$.

Remark 13.2. Obviously, $G \in \mathcal{X}^r$ if and only if $\bigcap_{\substack{N \trianglelefteq G \\ N \in \mathcal{X}}} N = 1$. If applicable, G is isomorphic to a subgroup of $\times_{\substack{N \trianglelefteq G \\ N \in \mathcal{X}}} G/N$. If $\{G_i : i \in I\}$ is an arbitrary family of groups, then $H \leq \times_{i \in I} G_i$ is called a *subdirect product*, if the projection of H onto each G_i is surjective.

Theorem 13.3. *Let $\mathcal{X}$ be a group class. Then $\mathcal{X}^r$ consists exactly of the subdirect products of $\mathcal{X}$-groups.*

Proof. According to Remark 13.2, every residual $\mathcal{X}$-group is a subdirect product of $\mathcal{X}$-groups. Conversely, let $H \leq \times_{i \in I} G_i$ be a subdirect product of $\mathcal{X}$-groups G_i. Let $\pi_i \colon H \to G_i$ be the projection and $K_i := \operatorname{Ker}(\pi_i) \trianglelefteq H$. By assumption, $H/K_i \cong G_i \in \mathcal{X}$ and $\bigcap_{i \in I} K_i = 1$. □

Example 13.4. If $\mathcal{X}$ is the group class of abelian groups, then $\mathcal{X}^r = \mathcal{X}$.

Theorem 13.5 (Iwasawa)**.** *Every free group is a residually finite p-group for every prime p.*

Proof. Let $a = x_1^{a_1} \dots x_n^{a_n} \in F_X \setminus \{1\}$ with $x_i \neq x_{i+1}$ and $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ as in Corollay 1.9. Let q be a power of p that does not divide $a_1 \dots a_n$. Let $E_{st} \in \mathbb{Z}/q\mathbb{Z}^{(n+1)\times(n+1)}$ be the matrix with a 1 at position (s,t) and zeros otherwise. For $x \in X$ let

$$\sigma(x) := \prod_{\substack{1 \leq i \leq n \\ x_i = x}} (1_{n+1} + E_{i,i+1}) \in \mathrm{GL}(n+1, \mathbb{Z}/q\mathbb{Z}).$$

Obviously $G := \langle \sigma(x) : x \in X \rangle$ consists of upper triangular matrices with ones on the main diagonal. Therefore G is a finite p-group. By the universal property, σ extends to a homomorphism $\widehat{\sigma} \colon F \to G$. Because of $E_{st}E_{uv} = \delta_{tu}E_{sv}$ and $x_i \neq x_{i+1}$, the factors of $\sigma(x)$ are pairwise commutable. This shows

$$\sigma(x_i)^{a_i} = \prod_{x_j = x_i} (1_{n+1} + E_{j,j+1})^{a_i} = \prod_{x_j = x_i} (1_{n+1} + a_i E_{j,j+1}) = 1_{n+1} + a_i \sum_{x_j = x_i} E_{j,j+1}.$$

From this it follows that

$$\widehat{\sigma}(a) = \sigma(x_1)^{a_1} \dots \sigma(x_n)^{a_n} = 1_{n+1} + a_1 \dots a_n E_{1,n+1} + \dots \neq 1_{n+1}. \qquad \square$$

Corollay 13.6 (Magnus)**.** *Every free group is residually nilpotent.*

Remark 13.7. For every non-abelian free group F, it holds that $F^{[\infty]} = \bigcap_{k=1}^{\infty} F^{[k]} = 1$ and $\mathrm{Z}(F) = 1$. The ascending and descending central series thus behave completely differently.

Lemma 13.8. *Let G be a finitely generated group. Then:*

(i) For every $n \in \mathbb{N}$, G possesses only finitely many subgroups with index n.

(ii) (Hall) *If $H \leq G$ with $|G : H| < \infty$, then there exists a characteristic subgroup $K \leq G$ with $K \leq H$ and $|G : K| < \infty$.*

Proof.

(i) Every subgroup $H \leq G$ of index n induces a homomorphism $G \to S_n$ with kernel H_G. Since G is finitely generated, there exist only finitely many such homomorphisms. Because of $H_G \leq H$, there are also only finitely many possibilities for H given H_G.

(ii) By (i), $\{\alpha(H) : \alpha \in \mathrm{Aut}(G)\}$ is finite. Then the characteristic subgroup $K := \bigcap_{\alpha \in \mathrm{Aut}(G)} \alpha(H)$ has finite index. $\square$

Theorem 13.9 (Baumslag)**.** *If G is finitely generated and residually finite, then $\mathrm{Aut}(G)$ is also residually finite.*

Proof. Let $\alpha \in \mathrm{Aut}(G) \setminus \{1\}$ and $g \in G$ with $\alpha(g) \neq g$. Then there exists $N \trianglelefteq G$ with $|G : N| < \infty$ and $\alpha(g)g^{-1} \notin N$. According to Lemma 13.8, the characteristic subgroup $M := \bigcap_{\beta \in \mathrm{Aut}(G)} \beta(N)$ has finite index in G. For $A := \mathrm{C}_{\mathrm{Aut}(G)}(G/M) \trianglelefteq \mathrm{Aut}(G)$, it therefore holds that $|\mathrm{Aut}(A)/A| \leq |\mathrm{Aut}(G/M)| < \infty$?!. Obviously $\alpha \notin A$. $\square$

Definition 13.10. A group G is called *Hopfian*, if G is not isomorphic to any proper factor group of G. This means that every epimorphism $G \to G$ is an automorphism.

Example 13.11. Obviously all finite groups and all simple groups are Hopfian. For dimension reasons, all vector spaces are also Hopfian. On the other hand, $C_2^{\mathbb{N}}$ is not Hopfian.

Theorem 13.12 (Mal'cev)**.** *Every finitely generated residually finite group is Hopfian.*

Proof. Let $\varphi \colon G \to G$ be a non-injective epimorphism. Let $g \in \mathrm{Ker}(\varphi) \setminus \{1\}$. Since G is residually finite, there exists an $N \trianglelefteq G$ with $g \notin N$ and $|G : N| < \infty$. Since G is finitely generated, there are only finitely many homomorphisms $\gamma_1, \dots, \gamma_n \colon G \to G/N$. Let γ_1 be the canonical epimorphism. From the surjectivity of φ it follows that $\{\gamma_1, \dots, \gamma_n\} = \{\gamma_1\varphi, \dots, \gamma_n\varphi\}$. Thus let $\gamma_1 = \gamma_k\varphi$. Then one obtains the contradiction $1 \neq gN = \gamma_1(g) = \gamma_k(\varphi(g)) = \gamma_k(1) = 1$. $\square$

Corollay 13.13. *Every free group with finite rank is Hopfian.*

Proof. Follows from Theorem 13.5. $\square$

Corollay 13.14. *Let F be free of rank $n < \infty$ and let $F = \langle Y \rangle$ with $|Y| = n$. Then F is free with respect to Y.*

Proof. Let $F = F_X$. An arbitrary bijection $X \to Y \subseteq F$ extends to an epimorphism $\sigma\colon F \to F$. According to Mal'cev, $\sigma \in \mathrm{Aut}(F)$. Let G be a group and $\tau\colon Y \to G$. Then the map $\tau\sigma_{|X}$ extends to a homomorphism $\gamma\colon F \to G$. Finally, $\gamma\sigma^{-1}\colon F \to G$ is an extension of τ. Because of $F = \langle Y \rangle$, this is the unique extension. Thus F satisfies the universal property with respect to Y. □

Theorem 13.15. *Every finitely generated residually nilpotent group is Hopfian.*

Proof. Let G be finitely generated and residually nilpotent. By assumption, $\bigcap_{k=1}^{\infty} G^{[k]} = 1$ and by Lemma 12.6, the factors $G^{[k]}/G^{[k+1]}$ are finitely generated abelian groups. Assume $G \cong G/N$ for some $1 \neq N \trianglelefteq G$. Let $k \in \mathbb{N}$ with $N \subseteq G^{[k]}$ and $N \nsubseteq G^{[k+1]}$. Then

$$G^{[k]}/G^{[k+1]} \cong (G/N)^{[k]}/(G/N)^{[k+1]} \cong G^{[k]}/G^{[k+1]}N \cong (G^{[k]}/G^{[k+1]})/(G^{[k+1]}N/G^{[k+1]}).$$

By the fundamental theorem of finitely generated abelian groups, $G^{[k]}/G^{[k+1]}$ is Hopfian. Contradiction. □

Theorem 13.16 (Hall)**.** *There exists a finitely generated solvable group that is not Hopfian.*

Proof. Let R be the ring of all numbers of the form $a2^b$ with $a, b \in \mathbb{Z}$ and N the group of upper 3×3 triangular matrices with ones on the main diagonal and elements from R. We set

$$u := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ v := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ w := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, let t be the diagonal matrix with diagonal $(1, 2, 1)$. Finally, we define $H := \langle t, N \rangle$.

We first show $H = \langle t, u, v \rangle$. Obviously $u^a = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $a \in \mathbb{Z}$. Because of

$$tu^a t^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a2^{-1} & 0 \\ 0 & 2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a2^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \langle t, u, v \rangle$ for all $a \in R$. Analogously, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \in \langle t, u, v \rangle$ for all $a \in R$. Finally,

$$\begin{aligned} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v^{-1} &= \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & a \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \langle t, u, v \rangle \end{aligned}$$

for all $a \in R$. For $a, b, c \in R$ we also have

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \langle t, u, v \rangle.$$

This shows $H = \langle N, t \rangle \subseteq \langle t, u, v \rangle \subseteq H$.

Because of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a2^{-1} & c \\ 0 & 2^{-1} & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a2^{-1} & c \\ 0 & 1 & b2 \\ 0 & 0 & 1 \end{pmatrix} \in N$$

we have $N \trianglelefteq H = N\langle t \rangle$ and obviously $N \cap \langle t \rangle = 1$. Now we consider the map $f \colon H \to H$ with

$$f\left(\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} t^d \right) = \begin{pmatrix} 1 & a & c2 \\ 0 & 1 & b2 \\ 0 & 0 & 1 \end{pmatrix} t^d$$

for $a, b, c \in R$ and $d \in \mathbb{Z}$. We show that f is an endomorphism:

$$f\left(\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} t^{d_1} \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} t^{d_2} \right) = f\left(\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 2^{-d_1} & c_2 \\ 0 & 1 & b_2 2^{d_1} \\ 0 & 0 & 1 \end{pmatrix} t^{d_1+d_2} \right)$$

$$= f\left(\begin{pmatrix} 1 & a_2 2^{-d_1} + a_1 & c_2 + a_1 b_2 2^{d_1} + c_1 \\ 0 & 1 & b_2 2^{d_1} + b_1 \\ 0 & 0 & 1 \end{pmatrix} t^{d_1+d_2} \right) = \begin{pmatrix} 1 & a_2 2^{-d_1} + a_1 & (c_2 + a_1 b_2 2^{d_1} + c_1)2 \\ 0 & 1 & (b_2 2^{d_1} + b_1)2 \\ 0 & 0 & 1 \end{pmatrix} t^{d_1+d_2}$$

$$= \begin{pmatrix} 1 & a_1 & c_1 2 \\ 0 & 1 & b_1 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 2^{-d_1} & c_2 2 \\ 0 & 1 & b_2 2^{d_1+1} \\ 0 & 0 & 1 \end{pmatrix} t^{d_1+d_2} = \begin{pmatrix} 1 & a_1 & c_1 2 \\ 0 & 1 & b_1 2 \\ 0 & 0 & 1 \end{pmatrix} t^{d_1} \begin{pmatrix} 1 & a_2 & c_2 2 \\ 0 & 1 & b_2 2 \\ 0 & 0 & 1 \end{pmatrix} t^{d_2}$$

$$= f\left(\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} t^{d_1} \right) f\left(\begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} t^{d_2} \right).$$

Obviously f is also bijective. Because of

$$uw = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = wu$$

$$vw = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = wv$$

we have $w \in \mathrm{Z}(H)$. In particular, $\langle w \rangle \trianglelefteq H$. The automorphism f now induces an isomorphism $H/\langle w \rangle \cong H/\langle w^2 \rangle$. Thus $G = H/\langle w^2 \rangle$ is a finitely generated group that is not Hopfian. Since $H/\mathrm{Z}(H)$ is abelian, H is solvable. □

Definition 13.17. Let $w = w(x_1, \ldots) \in W \subseteq F_X$. For a group G and $g_1, \ldots \in G$, let $w(g_1, \ldots) \in G$ be the element that arises by replacing x_i in w with g_i (only finitely many x_i appear in w). One calls

$$W(G) := \langle w(g_1, \ldots) : w \in W, g_1, \ldots \in G \rangle$$

the *verbal subgroup* with respect to W.

Example 13.18. For $W = \{[x, y]\}$, one obtains $W(G) = G'$.

Remark 13.19. For every homomorphism $f\colon G \to H$, it holds that $f(W(G)) \leq W(H)$, d. h. verbal subgroups are fully invariant.

Theorem 13.20 (Neumann)**.** *Every fully invariant subgroup of a free group is verbal.*

Proof. Let $F = F_X$ and $H \leq F$ be fully invariant. Let $w = w(x_1, \ldots) \in H$ and $y_1, \ldots \in F$ be arbitrary. Then there exists an endomorphism $\alpha\colon F \to F$ with $\alpha(x_i) = y_i$. It holds that $w(y_1, \ldots) = \alpha(w) \in H$. Thus $H = W(H)$ is verbal. □

Definition 13.21. Let again $W \subseteq F = F_X$. A normal subgroup N of a group G is called *W-marginal* if

$$w(g_1, \ldots, g_i a, g_{i+1}, \ldots) = w(g_1, \ldots)$$

for all $w \in W$, $g_1, \ldots \in G$, $i \in \mathbb{N}$ and $a \in N$. This is equivalent to

$$w(g_1, \ldots) = w(h_1, \ldots)$$

for all $g_1, h_1, \ldots \in G$ with $g_i \equiv h_i \pmod{N}$ for $i \in \mathbb{N}$. The subgroup generated by all W-marginal normal subgroups is called the *W-marginal subgroup* $W^*(G)$.

Remark 13.22. Obviously, $W^*(G)$ itself is W-marginal. For $W = \{[x, y]\}$, it holds that $W^*(G) = \mathrm{Z}(G)$, because

$$g \in W^*(G) \Rightarrow \forall h \in G : [g, h] = [1g, h] = [1, h] = 1 \Rightarrow g \in \mathrm{Z}(G) \Rightarrow g \in W^*(G).$$

Lemma 13.23. *It holds that $W(G) = 1$ if and only if $W^*(G) = G$.*

Proof. From $W(G) = 1$, it certainly follows that $W^*(G) = G$. Now let $W^*(G) = G$ and $g_1, \ldots \in G$. From $g_i \equiv 1 \pmod{G}$, it follows that $w(g_1, \ldots) = w(1, \ldots) = 1$. This shows $W(G) = 1$. □

Definition 13.24. A class of groups $\mathcal{X}$ is called a *variety*, if there exists $W \subseteq F_X$ with $\mathcal{X} = \{G : W(G) = 1\}$. If applicable, one writes $\mathcal{X} = \mathcal{X}(W)$.

Example 13.25. The abelian groups form the variety $\mathcal{X}([x, y])$. The elementary abelian p-groups form the variety $\mathcal{X}([x, y], x^p)$. The groups whose exponent divides n form the variety $\mathcal{X}(x^n)$ etc. The next theorem implies that the solvable groups of derived length $\leq n$ form a variety.

Theorem 13.26 (Birkhoff)**.** *A class of groups $\mathcal{X}$ is a variety if and only if $\mathcal{X}$ is closed under taking quotient groups and subdirect products.*

Proof. Every variety is closed with respect to subgroups, factor groups, and (sub)direct products. Now let $\mathcal{X}$ be a class of groups that is closed with respect to factor groups and subdirect products. Let $W \subseteq F_X$ be maximal with $W(G) = 1$ for all $G \in \mathcal{X}$. Then $\mathcal{X} \subseteq \mathcal{X}(W)$. Conversely, let $G \in \mathcal{X}(W)$. For each $w \in F_X \setminus W$, we choose $H(w) \in \mathcal{X}$ with $w(h_1, \ldots) \neq 1$ for certain $h_1, \ldots \in H(w)$. Let Y be a set that is at least as large as G and all $H(w)$ (for example $Y = G \cup \bigcup_w H(w)$). Then there exist

isomorphisms $F_Y/N \cong G$ and $F_Y/K(w) \cong H(w) \in \mathcal{X}$ with $w(y_1, \ldots) \notin K(w)$. For each $w \in F_Y \setminus N$, there exist $g_1, \ldots \in G$ with $w(g_1, \ldots) \neq 0$. Because of $G \in \mathcal{X}(W)$, it follows that $w \notin W$. We set

$$K := \bigcap_{w \in F_Y \setminus N} K(w) \trianglelefteq F_Y.$$

Then F_Y/K is a subdirect product of the $\mathcal{X}$-groups $H(w)$. By assumption, $F_Y/K \in \mathcal{X}$. For $w \in F_Y \setminus N$, we have $w \notin K(w)$, in particular $w \notin K$. Thus $K \subseteq N$ and $G \cong F_Y/N \cong (F_Y/K)/(N/K)$ is a factor group of an $\mathcal{X}$-group and thus $G \in \mathcal{X}$. This shows that $\mathcal{X} = \mathcal{X}(W)$ is a variety. □

Corollay 13.27. *If a class of groups is closed under the formation of factor groups and subdirect products, then it is also closed under the formation of subgroups.*

Definition 13.28. Let $\mathcal{X}$ be a variety. A group $F \in \mathcal{X}$ is called *$\mathcal{X}$-free with respect to $X \subseteq F$* if for every $\mathcal{X}$-group G and every mapping $\sigma\colon X \to G$, there exists exactly one homomorphism $\widehat{\sigma}\colon F \to G$ with $\widehat{\sigma}(x) = \sigma(x)$ for all $x \in X$.

Theorem 13.29. *Let $F = F_X$ be a free group and $\mathcal{X} = \mathcal{X}(W)$ a variety with $W \subseteq F$. Then $\overline{F} := F/W(F)$ is $\mathcal{X}$-free with respect to $\overline{X} := \{xW(F) : x \in X\}$ and every $\mathcal{X}$-free group with respect to $\overline{X}$ is isomorphic to $\overline{F}$.*

Proof. Let G be an $\mathcal{X}$-group and $\sigma\colon \overline{X} \to G$. We define $\tau\colon X \to G$ by $\tau(x) := \sigma(xW(F))$ for $x \in X$. Then there exists exactly one homomorphism $\widehat{\tau}\colon F \to G$ with $\widehat{\tau}(x) = \tau(x)$ for $x \in X$. Because of $G \in \mathcal{X}$, we have $\widehat{\tau}(w) = 1$ for all $w \in W(F)$. One obtains a homomorphism $\widehat{\sigma}\colon \overline{F} \to G$ with $\widehat{\sigma}(xW(F)) = \widehat{\tau}(x) = \tau(x) = \sigma(xW(F))$ for $x \in X$. Because of $\overline{F} = \langle \overline{X} \rangle$, $\widehat{\sigma}$ is uniquely determined by the images of $\overline{X}$.

Let H also be $\mathcal{X}$-free with respect to $\overline{X}$. Then $\mathrm{id}_{\overline{X}}$ can be extended to homomorphisms $\varphi\colon \overline{F} \to H$ and $\psi\colon H \to \overline{F}$. As usual, it follows that φ is an isomorphism. □

Remark 13.30. In contrast to free groups, there does not exist an $\mathcal{X}$-free group with respect to X for every set X. For $\mathcal{X} = \mathcal{X}(x) = \{1\}$, for example, every $\mathcal{X}$-free group is trivial.

Theorem 13.31. *Let $\mathcal{X}$ be a variety. Then every $\mathcal{X}$-group is isomorphic to a factor group of an $\mathcal{X}$-free group.*

Proof. Let $G \in \mathcal{X} = \mathcal{X}(W)$ with a minimal generating system X. Let $F := F_X$ and $\overline{X} := \{xW(F) : x \in X\}$ as in Theorem 13.29. Let $x, y \in X$ with $x \equiv y \pmod{W(F)}$. Then $y^{-1}x \in W(F)$. From $W(G) = 1$ it follows that $x = y$. Therefore $|\overline{X}| = |X|$ holds and there exists a bijection $\sigma\colon \overline{X} \to X \subseteq G$, which extends to an epimorphism $\overline{F} \to G$. The claim follows from Theorem 13.29. □

Theorem 13.32 (Engel)**.** *Let $x_1, \ldots, x_k \in X$ with $x_{k-1} \neq x_k$. Then every finite group $G \in \mathcal{X}([x_1, \ldots, x_k])$ is nilpotent.*

Proof. Let G be a minimal counterexample. Then every proper subgroup of G is nilpotent. According to Schmidt, $G = P \rtimes Q$ with $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Assume $\Phi(P) \neq 1$. Then $G/\Phi(P)$ is also nilpotent. Because of $\Phi(P) \leq \Phi(G)$, then $G/\Phi(G)$ and G would be nilpotent. Thus $\Phi(P) = 1$, d. h. P is elementary abelian. For $x \in P$ and $y = y_1 = \ldots = y_{k-1} \in Q \setminus \{1\}$, $[y_1, \ldots, y_{k-1}, x] = 1$ holds by assumption. Since P is abelian, there exists $\alpha \in \mathrm{End}(P)$ with $\alpha(g) = [y, g]$ for $g \in P$. Inductively one obtains $\alpha^{k-1}(x) = [y_1, \ldots, y_{k-1}, x] = 1$. If $\beta \in \mathrm{Aut}(P)$ is the conjugation by y, then $\alpha = \beta - 1 \in \mathrm{End}(P)$ holds. Let $m \in \mathbb{N}$ with $p^m \geq k$. Since P is elementary abelian, it follows

$$0 = \alpha^{p^m} = \sum_{i=1}^{p^m} \binom{p^m}{i} (-1)^i \beta^{p^m - i} = \beta^{p^m} - 1$$

(for $p = 2$, $-1 = 1$). Thus $y^{p^m} \in \mathrm{C}_G(P)$ holds and $y \in \mathrm{C}_G(P)$ because of $p \neq q$. But then G would be nilpotent. □

Corollay 13.33 (Zorn)**.** *Let G be a finite group and $k \in \mathbb{N}$ with $[\underbrace{g, \ldots, g}_{k}, h] = 1$ for all $g, h \in G$, then G is nilpotent.*

Remark 13.34. If even $[g_1, \ldots, g_k] = 1$ for all $g_1, \ldots, g_k \in G$, then G is nilpotent with nilpotency class $\leq k$. See also Lemma 12.7.

14 p-groups

Definition 14.1. Let $X = \{x_1, \ldots\}$ and $\mathcal{X} = \mathcal{X}(x^{p^2}, [x, y]^p, [x, y, z] : x, y, z \in F_X)$. Let G_r be the $\mathcal{X}$-free group of rank $r \geq 1$ from Theorem 13.29. As usual, we identify x_i with the corresponding coset in G_r.

Lemma 14.2. *The variety $\mathcal{X}$ consists of all p-groups G with $\Phi(G) \leq \mathrm{Z}(G)$ and $\Phi(\Phi(G)) = 1$.*

Proof. Let G be a p-group with $\Phi(G) \leq \mathrm{Z}(G)$ and $\Phi(\Phi(G)) = 1$. For $x, y, z \in G$, it holds that $x^p \in \Phi(G)$ and $x^{p^2} = 1$. Furthermore, $[x, y] \in G' \leq \Phi(G)$ and $[x, y]^p = 1$ follows. Finally, $[y, z] \in \Phi(G) \leq \mathrm{Z}(G)$ and therefore $[x, y, z] = [x, [y, z]] = 1$.

Conversely, let $G \in \mathcal{X}$ and $N := \langle x^p, [x, y] : x, y \in G \rangle$. Then G/N is elementary abelian (possibly infinite) and $\Phi(G) \leq N$. Because of $[x^p, y] = [x, y]^p = 1$ and $[x, y, z] = 1$, it follows that $N \leq \mathrm{Z}(G)$ and N is elementary abelian, i. e. $\Phi(\Phi(G)) \leq \Phi(N) = 1$. □

Lemma 14.3. *Let $G \in \mathcal{X}$ and $y_1, \ldots, y_r \in G$. Then there exists a homomorphism $\varphi\colon G_r \to G$ with $\varphi(x_i) = y_i$ for $i = 1, \ldots, r$.*

Proof. Follows from Theorem 13.31 or Theorem 1.15 (since $y_1, \ldots, y_r$ is not necessarily a generating set of G, one only obtains a homomorphism instead of an epimorphism). □

Lemma 14.4. *The following holds:*

(i) $\Phi(G_r)$ is elementary abelian of rank $r(r+1)/2$.

(ii) $G_r/\Phi(G_r)$ is elementary abelian of rank r.

(iii) If $\alpha \in \mathrm{Aut}(G_r)$ acts trivially on $G_r/\Phi(G_r)$, then it also acts trivially on $\Phi(G_r)$.

Proof.

(i) Because $[x_i, x_j x_k] = [x_i, x_j] \cdot {}^{x_j}[x_i, x_k] = [x_i, x_j][x_i, x_k]$ for $1 \le i, j, k \le r$, $G_r' = \langle [x_i, x_j] : 1 \le i < j \le r \rangle$ is elementary abelian of rank at most $r(r-1)/2$. Because $(x_i x_j)^p \equiv x_i^p x_j^p \pmod{G_r'}$, $G_r^p G_r' = \langle x_i^p, [x_i, x_j] \rangle$ is elementary abelian of rank at most $r(r-1)/2 + r = r(r+1)/2$. Since $G/G_r^p G_r'$ is also elementary abelian of rank at most r, G is a finite p-group and $\Phi(G_r) = G_r^p G_r' \le \mathrm{Z}(G_r)$.

Suppose there is a relation of the form

$$g := \prod_{i=1}^{r} x_i^{pa_i} \prod_{1 \le i < j \le r} [x_i, x_j]^{b_{ij}} = 1$$

with $0 \le a_i, b_{ij} \le p-1$. For $\langle h \rangle \cong C_{p^2}$ and $1 \le i \le r$, there exists by Lemma 14.3 a homomorphism $\varphi \colon G_r \to \langle h \rangle$ with $\varphi(x_i) = h$ and $\varphi(x_j) = 1$ for $j \ne i$. It follows that $1 = \varphi(g) = h^{pa_i}$ and $a_i = 0$ for $i = 1, \dots, r$. For $1 \le i < j \le r$, let analogously $\langle h_i, h_j \rangle \cong p_+^{1+2}$. Again, there exists a homomorphism $\varphi \colon G_r \to \langle h_i, h_j \rangle$ with $\varphi(x_i) = h_i$, $\varphi(x_j) = h_j$ and $\varphi(x_k) = 1$ for $i \ne k \ne j$. From $1 = \varphi(g) = [h_i, h_j]^{b_{ij}}$ it follows that $b_{ij} = 0$. This shows that $\Phi(G_r)$ has rank $r(r+1)/2$.

(ii) Let $0 \le a_1, \dots, a_r \le p-1$ with $g = x_1^{a_1} \dots x_r^{a_r} \in \Phi(G_r)$. As in (i), there exists a homomorphism $\varphi \colon G_r \to \langle h \rangle \cong C_{p^2}$ with $h^{a_i} = \varphi(g) \in \Phi(\langle h \rangle) = \langle h^p \rangle$. This shows $g = 1$ and the assertion follows.

(iii) Let $\alpha \in \mathrm{Aut}(G_r)$ be trivial on $G_r/\Phi(G_r)$. Then there exist $h_1, \dots, h_r \in \Phi(G_r)$ with $\alpha(x_i) = x_i h_i$ for $i = 1, \dots, r$. Because of (i), $\alpha(x_i^p) = \alpha(x_i)^p = (x_i h_i)^p = x_i^p$ and $\alpha([x_i, x_j]) = [x_i h_i, x_j h_j] = [x_i, x_j]$ for $1 \le i < j \le r$. □

Lemma 14.5. *Let $N, M \le \Phi(G_r)$. $G_r/N \cong G_r/M$ holds if and only if there exists an $\alpha \in \mathrm{Aut}(G_r)$ with $\alpha(N) = M$.*

Proof. Every $\alpha \in \mathrm{Aut}(G_r)$ with $\alpha(N) = M$ induces an isomorphism $G_r/N \cong G_r/M$. Conversely, let an isomorphism $\alpha' \colon G_r/N \to G_r/M$ be given. Choose $y_1, \dots, y_r \in G_r$ with $\alpha'(x_i N) = y_i M$ for $i = 1, \dots, r$. By Lemma 14.3 and Lemma 14.4, there exists a homomorphism $\alpha \colon G_r \to G_r$ with $\alpha(x_i) = y_i$ for $i = 1, \dots, r$. As is well known,

$$G_r = \langle y_1, \dots, y_r \rangle M = \langle y_1, \dots, y_r \rangle \Phi(G_r) = \langle y_1, \dots, y_r \rangle.$$

Therefore, α is surjective and an isomorphism. Because $\alpha(x_i)M = y_i M = \alpha'(x_i N)$, it holds that $\alpha(g)M = \alpha'(gN)$ for all $g \in G_r$. In this case,

$$g \in N \iff \alpha'(gN) = 1 \iff \alpha(g)M = 1 \iff \alpha(g) \in M.$$

This shows $\alpha(N) = M$. □

Lemma 14.6. *The number of d-dimensional subspaces of $\mathbb{F}_p^n$ is*

$$\binom{n}{d}_p = \frac{(p^n - 1)(p^n - p) \dots (p^n - p^{d-1})}{(p^d - 1)(p^d - p) \dots (p^d - p^{d-1})}$$

and it holds that $p^{d(n-d)} \le \binom{n}{d}_p \le p^{d(n-d+1)}$.

Proof. The formula is known from combinatorics. The lower bound follows from $p^{n-d} \leq \frac{p^n-p^i}{p^d-p^i}$ for $0 \leq i \leq d-1$. On the other hand, $p^n + p^{n-d+i+1} \leq 2p^n \leq p^{n+1} + p^i$ and this yields $\frac{p^n-p^i}{p^d-p^i} \leq p^{n-d+1}$. □

Lemma 14.7. *Let $r, s \in \mathbb{N}$ with $1 \leq s \leq r(r+1)/2$. Then there exist at least $p^{rs(r+1)/2-r^2-s^2}$ non-isomorphic groups of order p^{r+s}.*

Proof. Let $\mathcal{N}$ be the set of normal subgroups $N \trianglelefteq \Phi(G_r)$ with $|\Phi(G_r) : N| = p^s$. According to Lemma 14.4 and Lemma 14.6, $|\mathcal{N}| \geq p^{rs(r+1)/2-s^2}$ holds. For $N \in \mathcal{N}$, $|G_r/N| = p^{r+s}$ holds. According to Lemma 14.5, it suffices to estimate the number of orbits of $\mathrm{Aut}(G_r)$ on $\mathcal{N}$. Let $\Gamma\colon \mathrm{Aut}(G_r) \to \mathrm{Aut}(G_r/\Phi(G_r))$ be the canonical homomorphism. According to Lemma 14.4, the automorphisms in $\mathrm{Ker}(\Gamma)$ act trivially on $\mathcal{N}$. On the other hand,

$$|\mathrm{Aut}(G_r/\Phi(G_r))| = |\mathrm{GL}(r,p)| = (p^r-1)(p^{r-1}-1)\dots(p^r-p^{r-1}) \leq p^{r^2}.$$

Each orbit of $\mathrm{Aut}(G_r)$ on $\mathcal{N}$ thus has length at most p^{r^2}. The number of orbits is therefore at least $p^{rs(r+1)/2-r^2-s^2}$. □

Theorem 14.8 (Higman)**.** *Let $f(p^n)$ be the number of non-isomorphic groups of order p^n for a prime p. Then*

$$\boxed{p^{\frac{2}{27}n^2(n-6)} \leq f(p^n) \leq p^{\frac{1}{6}(n^3-n)}.}$$

Proof. For the lower bound, we can assume $n \geq 6$. Let

$$s := \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3, \\ (n+2)/3 & \text{if } n \equiv 1 \pmod 3, \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

and $r = n - s$. Then $s \leq r \leq r(r+1)/2$ and

$$\begin{aligned} rs(r+1)/2 - r^2 - s^2 &= \begin{cases} \frac{1}{27}n^2(2n+3) - \frac{4}{9}n^2 - \frac{1}{9}n^2 & \text{if } n \equiv 0 \pmod 3 \\ \frac{1}{27}(n+2)(n-1)(2n+1) - \frac{4}{9}(n-1)^2 - \frac{1}{9}(n+2)^2 & \text{if } n \equiv 1 \pmod 3 \\ \frac{1}{27}(n+1)(2n-1)(n+1) - \frac{1}{9}(2n-1)^2 - \frac{1}{9}(n+1)^2 & \text{if } n \equiv 2 \pmod 3 \end{cases} \\ &= \begin{cases} \frac{2}{27}n^2(n-6) & \text{if } n \equiv 0 \pmod 3 \\ \frac{2}{27}n^2(n-6) + \frac{1}{3}n - \frac{26}{27} & \text{if } n \equiv 1 \pmod 3 \\ \frac{2}{27}n^2(n-6) + \frac{2}{9}n - \frac{7}{27} & \text{if } n \equiv 2 \pmod 3 \end{cases} \\ &\geq \frac{2}{27}n^2(n-6). \end{aligned}$$

By Lemma 14.7, it follows that $f(p^n) \geq p^{\frac{2}{27}n^2(n-6)}$.

For the upper bound, let G be a group of order p^n with chief series $G = H_0 > H_1 > \ldots > H_n = 1$. We choose $g_i \in H_{i-1} \setminus H_i$ for $i = 1, \dots, n$. Every element in G can then be uniquely written in the form $g = g_1^{a_1} \dots g_n^{a_n}$ with $0 \leq a_1, \dots, a_n \leq p-1$. Here $g \in H_i$ holds if and only if $a_1 = \ldots = a_i = 0$. Let $0 \leq b_{ij} \leq p-1$ with

$$g_i^p = g_{i+1}^{b_{i,i+1}} \dots g_n^{b_{i,n}}. \tag{14.1}$$

Because $H_{j-1}/H_j \leq \mathrm{Z}(G/H_j)$, it follows that $[g_i, g_j] \in H_j$ for $1 \leq i < j \leq n$. Therefore there exist $0 \leq c_{ij} \leq p-1$ with

$$[g_i, g_j] = g_{j+1}^{c_{i,j+1}} \dots g_n^{c_{i,n}} \qquad (i < j). \tag{14.2}$$

We now show that G is uniquely determined by the relations (14.1) and (14.2). For this, one must bring a product $g_1^{a_1} \dots g_n^{a_n} g_1^{a'_1} \dots g_n^{a'_n}$ into the form $g_1^{a''_1} \dots g_n^{a''_n}$. This is clear for $n = 1$. Let $n \geq 2$. Using (14.2), one can shift $g_1^{a'_1}$ to the left by inserting terms of the form $g_j^{c_{1,j}}$ with $j \geq 2$. Subsequently, one obtains $g_1^{a_1+a'_1} h$ with $h \in H_1$. Using (14.1), one can replace $a_1 + a'_1$ by a''_1 by inserting further terms of the form $g_i^{b_{ij}}$ with $i \geq 2$. The claim now follows by induction. The number of non-isomorphic groups of order p^n is thus bounded by the choice of the parameters b_{ij} and c_{ij}. For the choice of the b_{ij}, there are $p^{n(n-1)/2}$ possibilities. For the choice of the c_{ij}, there are p^α possibilities, where

$$\alpha = \sum_{i=1}^{n-2} \binom{n-i}{2} = \binom{n}{3} = \frac{n^3 - 3n^2 + 2n}{6}.$$

(One counts the number of 3-element subsets of $\{1, \dots, n\}$ with a given maximum.) In total, there are at most $p^{(n^3-n)/6}$ groups of order p^n. □

Remark 14.9. Sims-Newman-Seeley have proven the stronger estimate $f(p^n) \leq p^{\frac{2}{27}n^3 + O(n^{5/2})}$. Since we only used factor groups of G_r to prove the lower estimate, it follows: Almost all p-groups G possess a normal subgroup $N \leq \mathrm{Z}(G)$ such that N and G/N are elementary abelian. In particular, almost all p-groups have nilpotency class 2.

Definition 14.10. For finite sets $A, B \subseteq \mathbb{N}$, let $A \prec B$ if $|A| < |B|$ or if $|A| = |B|$ and A is lexicographically smaller than B (i. e. $\min(A \cup B) \setminus (A \cap B) \in A$).

Remark 14.11. Obviously, $\prec$ is a total order on the set of finite subsets of $\mathbb{N}$. From $A \subseteq B$ follows $A \prec B$.

Lemma 14.12 (Hall's Collector Process)**.** *Let $X = \{x_1, \dots, x_n\} = X_1 \dot\cup \dots \dot\cup X_m$ and $F := F_X$. For $I \subseteq \{1, \dots, m\}$ let*

$$C_I := F^{[|I|]} \cap \bigcap_{i \in I} \langle X_i \rangle^F \cap \langle \bigcup_{i \in I} X_i \rangle \leq F.$$

Then there exist $c_I \in C_I$ with

$$x_1 \dots x_n = \prod_{I \subseteq \{1,\dots,m\}} c_I,$$

where the subsets I are traversed according to $\prec$.

Proof. For $J \subseteq \{1, \dots, m\}$ we show

$$x_1 \dots x_n = \prod_{I \prec J} c_I \cdot y_1 \dots y_n$$

with $y_1, \dots, y_n \in C_K$ for some $J \preceq K$. The assertion then follows with $J = \{1, \dots, m\}$ by setting $c_J = y_1 \dots y_n$. For $J = \varnothing$, the product over $I \prec J$ is empty. For $x_i \in X_j \leq \langle X_i \rangle = C_{\{j\}}$, one can choose $y_i = x_i$. Now let the assertion already be proven for J and let J' be the successor of J w.r.t. $\prec$. We can assume that at least one y_i lies in C_J, because otherwise $y_i \in C_K$ with $J' \preceq K$ for all i. Let i be minimal in this case. In the case $i > 1$, we write $y_{i-1} y_i = y_i y_{i-1} z$ with $z := [y_{i-1}^{-1}, y_i^{-1}]$ and $y_{i-1} \in C_K \subseteq F^{[|K|]}$, where $J \prec K$. Then it holds that

- $z \in [F^{[|K|]}, F^{[|J|]}] \leq F^{[|K|+|J|}} \leq F^{[|J \cap K|]}$.

- $z \in \langle X_j \rangle^F$ for all $j \in J \cup K$.
- $z \in \langle \bigcup_{j \in J \cup K} X_j \rangle$.

This shows $z \in C_{J \cup K}$ with $J \prec J \cup K$. In finitely many steps, we can move all $y_i \in C_J$ to the left in this way. We call the product of these y_i as c_J. For the remaining factors, $y_j \in C_K$ with $J' \preceq K$ now holds as desired. □

Theorem 14.13 (Hall-Petrescu formula)**.** *For every group G and $x, y \in G$, there exist uniquely determined elements $c_i \in G^{[i]}$ such that for all $n \in \mathbb{N}$*

$$\boxed{x^n y^n = (xy)^n c_2^{\binom{n}{2}} c_3^{\binom{n}{3}} \dots c_n^{\binom{n}{n}}.}$$

Proof. Wlog. let $G := \langle x, y \rangle$. The uniqueness of the c_i follows inductively:

$$c_2 = (xy)^{-2} x^2 y^2, \qquad c_n = c_{n-1}^{-\binom{n}{n-1}} \dots c_2^{-\binom{n}{2}} (xy)^{-n} x^n y^n.$$

Let $N \in \mathbb{N}$ be arbitrary. We show that the formula holds for all $n \leq N$. Due to uniqueness, it then holds for all $n \in \mathbb{N}$. Let $X_i := \{x_i, x_{i+N}\}$ and $X = X_1 \dot\cup \dots \dot\cup X_N = \{x_1, \dots, x_{2N}\}$. Let $F := F_X$. For $I \subseteq \{1, \dots, N\}$, let $\mu_I \colon F \to F$ be the endomorphism with

$$\mu_I(x_i) = \begin{cases} x_i & \text{if } i \in I \vee i - n \in I, \\ 1 & \text{otherwise.} \end{cases}.$$

Let $x_1 \dots x_{2N} = \prod c_I$ as in Lemma 14.12. For $I \subseteq J$, $\mu_J(c_I) = c_I$ holds, because $C_I \subseteq \langle \bigcup_{i \in I} X_i \rangle$. For $I \nsubseteq J$, however, $\mu_J(c_I) = 1$ holds, because for $i \in I \setminus J$ we have $C_I \subseteq \langle x_i \rangle^F = \langle x_i, x_{i+N} \rangle^F \subseteq \operatorname{Ker}(\mu_J)$. According to Lemma 14.12, it thus holds that

$$\prod_{j \in J} x_j \prod_{j \in J} x_{j+N} = \mu_J(x_1 \dots x_{2N}) = \prod_{I \subseteq J} c_I,$$

where the sets I are ordered with respect to $\prec$.

Let $\varphi \colon F \to G$ be a homomorphism with $\varphi(x_i) = x$ and $\varphi(x_{i+N}) = y$ for $i = 1, \dots, N$. For $|J| = n$ we obtain

$$x^n y^n = \varphi(\mu_J(x_1 \dots x_{2N})) = \prod_{I \subseteq J} \varphi(c_I).$$

We claim that $\varphi(c_I)$ only depends on $|I|$. This is clear for $n = 0$ with $c_\varnothing = 1$. For $n = 1$, one obtains $\varphi(c_I) = xy$ for all single-element sets I. Now let $n := |J| = |J'|$ and

$$\Big(\prod_{I \subsetneq J} \varphi(c_I)\Big)\varphi(c_J) = \prod_{I \subseteq J} \varphi(c_I) = x^n y^n = \prod_{I' \subseteq J'} \varphi(c'_I) = \Big(\prod_{I' \subsetneq J'} \varphi(c'_I)\Big)\varphi(c_{J'}).$$

On both sides, the I are sorted ascendingly by size. For $l := |I| = |I'| < n$, $\varphi(c_I) = \varphi(c_{I'})$ already holds by induction. The number of these factors is $\binom{n}{l}$ on both sides. Therefore $\varphi(c_J) = \varphi(c_{J'})$.

We set $c_l := \varphi(c_{\{1,\dots,l\}})$ for $l = 1, \dots, N$. Then $c_l \in \varphi(F^{[l]}) \subseteq G^{[l]}$ and

$$x^n y^n = \prod_{l=1}^{n} c_l^{\binom{n}{l}} = c_1^n \prod_{l=2}^{n} c_l^{\binom{n}{l}} = (xy)^n \prod_{l=1}^{n} c_l^{\binom{n}{l}}.$$ □

Example 14.14. It holds that $c_2 = (xy)^{-2}x^2y^2 = y^{-1}x^{-1}y^{-1}xy^2 = [y^{-1}x^{-1}, y^{-1}]$. In the case $\langle x, y\rangle^{[3]} = 1$, one obtains the formula $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$ known from GT-Exercise 18.

Lemma 14.15. *Let p be a prime and G nilpotent with class $k < p$. For all $x, y \in G$ and $q = p^n$, the following hold:*

(i) *There exists a $z \in G'$ such that $x^q y^q = (xy)^q z^q$.*

(ii) $x^q = y^q \iff (xy^{-1})^q = 1$.

(iii) $\Omega_q(G) := \{x \in G : x^q = 1\} \trianglelefteq G$.

(iv) $\mho_q(G) := \{x^q : x \in G\} \trianglelefteq G$.

(v) $|G| = |\Omega_q(G)||\mho_q(G)|$.

(vi) $\mho_q(G)' \in \mho_{q^2}(G')$.

Proof. Induction on k: In the case $k \leq 1$, G is abelian and all statements follow easily. Let $k \geq 2$ and wlog. $G = \langle x, y\rangle$.

(i) By assumption, $G^{[p]} \leq G^{[k+1]} = 1$. By the Hall-Petrescu formula, there exist $c_i \in G^{[i]} \leq G'$ with

$$x^q y^q = (xy)^q c_2^{\binom{q}{2}} \dots c_{p-1}^{\binom{q}{p-1}}.$$

One easily sees that $\binom{q}{i}$ is divisible by q for $0 < i < p$. By induction, (iv) holds for G'. This shows $c_2^{\binom{q}{2}} \dots c_{p-1}^{\binom{q}{p-1}} \in \mho_q(G')$.

(ii) Let $x^q = y^q$. Then

$$x^q = yx^q y^{-1} = (yxy^{-1})^q.$$

Let $H := \langle x, yxy^{-1}\rangle \leq G$. Because of

$$[x, yxy^{-1}] = [x, yxy^{-1}x^{-1}] = [x, y, x] \in G^{[3]}$$

it holds that $H' = \langle [x, yxy^{-1}]\rangle^H \leq G^{[3]}$. Therefore H has nilpotency class $< k$. By induction, $[x, y]^q = (x(yxy^{-1})^{-1})^q = 1$. Since G' also has nilpotency class $< k$, it holds that

$$G' = \langle [x, y]\rangle^G = \langle g[x, y]g^{-1} : g \in G\rangle \leq \Omega_q(G') = G',$$

i. e. $\exp(G') \leq q$. From (i) it follows that $(xy^{-1})^q = x^q y^{-q} = 1$.

Conversely, let $(xy^{-1})^q = 1$. Then also $(y^{-1}x)^q = x^{-1}(xy^{-1})^q x = 1$. By the first part of the equivalence, it follows that $[x, y^{-1}]^q = (xy^{-1}x^{-1}y)^q = 1$. As before, one obtains $\exp(G') \leq q$. By (i), $x^q y^{-q} = (xy^{-1})^q = 1$ and $x^q = y^q$.

(iii) For $x, y \in \Omega_q(G)$, it holds that $x^q = 1 = y^q$ and $(xy^{-1})^q = 1$ by (ii). This shows $xy^{-1} \in \Omega_q(G)$ and $\Omega_q(G) \leq G$. Obviously, $\Omega_q(G)$ is a normal subgroup of G.

(iv) For $x^q, y^q \in \mho_q(G)$, there exists by (i) a $z \in G'$ with $x^q y^{-q} = (xy^{-1})^q z^q$. For $H := \langle xy^{-1}, z\rangle$, it holds that $[xy^{-1}, z] \in G^{[3]}$ and $H' = \langle [xy^{-1}, z]\rangle^H \leq G^{[3]}$. In particular, H has nilpotency class $< k$. By induction it holds that $(xy^{-1})^q z^q \in \mho_q(H) \leq \mho_q(G)$ and $\mho_q(G) \leq G$. Obviously, $\mho_q(G)$ is normal in G.

(v) The map $f\colon G \to \mho_q(G)$, $x \mapsto x^q$ is surjective (but in general not a homomorphism). By (ii), $f(x) = f(y)$ holds if and only if $xy^{-1} \in \Omega_q(G)$. Therefore $f^{-1}(x^q) = x\Omega_q(G)$ and $|f^{-1}(x)| = |\Omega_q(G)|$ for all $x \in G$. If $\Omega_q(G)$ is infinite, then so is G. Otherwise, $|G : \Omega_q(G)| = |\mho_q(G)|$.

(vi) Let $x^q, y^q \in \mho_q(G)$. We apply (ii) to $\overline{G'} := G'/\mho_{q^2}(G')$:

$$[x^q, y^q] = x^q(y^q x y^{-q})^{-q} \in \mho_{q^2}(G') \iff (xy^q x^{-1} y^{-q})^q \in \mho_{q^2}(G') \iff xy^q x^{-1} y^{-q} \in \Omega_q(\overline{G'}).$$

Now we apply (ii) to $\overline{G'}/\Omega_q(\overline{G'})$:

$$xy^q x^{-1} y^{-q} = (xyx^{-1})^q y^{-q} \in \Omega_q(\overline{G'}) \iff [x,y]^q \in \Omega_q(\overline{G'}) \iff [x,y]^{q^2} \in \mho_{q^2}(G').$$

The statement on the right side is obviously true. □

Example 14.16. The statements in Lemma 14.15 do not hold in general for groups with nilpotency class $\geq p$. For example, $\Omega_2(D_8)$ cannot be a subgroup of D_8 because of $|\Omega_2(D_8)| = 6$. For

$$G := \langle x, y \mid x^4 = y^4 = 1,\ yxy^{-1} = x^{-1} \rangle \cong C_4 \rtimes C_4.$$

$\mho_2(G) = \{1, x^2, y^2\}$ is not a subgroup of G.

Remark 14.17. The next theorem shows that p-groups with "small" nilpotency class behave approximately like abelian groups.

Theorem 14.18 (Groves)**.** *Let P be a p-group with nilpotency class $< p$. Then there exists an operation $P \times P \to P$, $(x, y) \mapsto x + y$ with the following properties:*

(i) $P_+ := (P, +)$ is an abelian group.

(ii) The orders of x in P and P_+ are equal.

(iii) For all $x, y \in P$, $x + y \in \langle x, y \rangle$ holds.

(iv) Every automorphism of P is also an automorphism of P_+.

(v) Every subgroup of P is also a subgroup of P_+.

Proof. Let $P = \langle x_1, \ldots, x_n \rangle$ and $F := F_n/F_n^{[p]}$ be the free nilpotent group with rank n and nilpotency class $p - 1$. Then there exists an epimorphism $f\colon F \to P$, $x \mapsto \overline{x}$. Let $q = \exp(P')$. For all $x, y \in F$, there exists by Lemma 14.15 (applied to $\langle x, y \rangle$) an $s = s(x, y) \in \langle x, y \rangle$ with $x^q y^q = s^q$. Let also $t \in F$ with $s^q = t^q$. Then $(st^{-1})^q = 1$ follows from Lemma 14.15. Since F is torsion-free (Corollay 12.34), it follows that $s = t$, i. e. s is uniquely determined by x, y. We define $\overline{x} + \overline{y} := \overline{s}$.

(i) For $x, y, z \in F$ we have

$$s(s(x,y),z)^q = s(x,y)^q z^q = (x^q y^q) z^q = x^q (y^q z^q) = x^q s(y,z)^q = s(x, s(y,z))^q.$$

As above, it follows that $s(s(x, y), z) = s(x, s(y, z))$. This shows that $+$ is associative on P. Because of $s(x, 1) = 1 = s(1, x)$, $\overline{1}$ is neutral with respect to $+$. Because of $s(x, x^{-1}) = 1$, $\overline{x^{-1}}$ is inverse to $\overline{x}$ with respect to $+$.

According to Lemma 14.15, $s(x,y)^q s(y,x)^{-q} = [x^q, y^q] \in \mho_{q^2}(F')$ and $(s(x,y)s(y,x)^{-1})^q \in \mho_{q^2}(F')$. Let $z \in F'$ with $(s(x,y)s(y,x)^{-1})^q = z^{q^2} = (z^q)^q$. From Lemma 14.15 it follows that

$(s(x,y)s(y,x)^{-1}z^{-q})^q = 1$. Since F is torsion-free, one obtains $s(x,y)s(y,x)^{-1} = z^p \in \mho_q(F')$. Because of $\mho_q(P') = 1$, it follows that

$$\overline{x} + \overline{y} = \overline{s(x,y)} = \overline{s(y,s)} = \overline{y} + \overline{x}$$

for all $x, y \in P$, i. e. P_+ is an abelian group.

(ii) Obviously $\overline{x} + \overline{x} = \overline{s(x,x)} = \overline{x^2}$ and inductively $k \cdot \overline{x} = \overline{x^k}$ for all $k \in \mathbb{N}$.

(iii) Follows from $s(x,y) \in \langle x, y\rangle$.

(iv) Let $\alpha \in \mathrm{Aut}(P)$. According to (iii), $\overline{x} + \overline{y}$ is a word in $\overline{x}$ and $\overline{y}$. Therefore $\alpha(\overline{x} + \overline{y})$ is the corresponding word in $\alpha(\overline{x})$ and $\alpha(\overline{y})$. This shows $\alpha(\overline{x} + \overline{y}) = \alpha(\overline{x}) + \alpha(\overline{y})$.

(v) Let $Q \leq P$, $H := f^{-1}(Q) \leq F$ and $\overline{x}, \overline{y} \in Q$. Then $s(x, y^{-1}) \in \langle x, y^{-1}\rangle \leq H$ and $\overline{x} - \overline{y} \in Q$. This shows $Q \leq P_+$. □

Remark 14.19.

(i) The isomorphism type of P_+ can be easily determined from $|\Omega_q(P)| = |\Omega_q(P_+)|$ for $q = p, p^2, \ldots$.

(ii) Let $p > 2$ and P be a p-group with nilpotency class 2. Let $q := \exp(P')$. In the free group with nilpotency class 2, the following holds:

$$x^q y^q = (xy)^q [y,x]^{-\binom{q}{2}} = (xy[y,x]^{\frac{1-q}{2}})^q.$$

according to Example 14.14. Since the map $P \to P$, $x \mapsto x^2$ is bijective, every $x \in P$ has exactly one "root" $\sqrt{x} \in P$ with $\sqrt{x}^2 = x$. As in the proof of Theorem 14.18, one obtains an abelian group structure on P via

$$x + y := xy[y,x]^{\frac{1-q}{2}} = xy\sqrt{[y,x]}$$

This special case was first constructed by Baer.

(iii) For p-groups with nilpotency class $3 < p$, the formula is

$$x + y := xy\sqrt{[y^{-1}, x^{-1}]}\ \sqrt[12]{[x,y,x]}\ \sqrt[12]{[y,x,y]}$$

(without proof). In general, these terms are obtained from the *Baker-Campbell-Hausdorff formula*.

(iv) Theorem 14.18 already holds if every subgroup of G generated by three elements has nilpotency class $< p$ (three generators are necessary to prove the associative law).

(v) The *Lazard correspondence* provides a Lie ring structure on p-groups with nilpotency class $< p$. Here, P_+ is the additive group of this Lie ring.

Example 14.20. All p-groups P with $|P| \leq p^p$ have nilpotency class $< p$ and therefore satisfy the assumption of Theorem 14.18. For $P = \texttt{SmallGroup}(5^5, 21)$, $|\Omega_5(P)| = 5^3$ and $|\Omega_{25}(P)| = 5^4$ holds. Therefore $P_+ \cong C_{5^3} \times C_5^2$. One can show that P_+ has exactly 56 subgroups of order 25. According to Theorem 14.18, P has at most as many subgroups (actually there are only 16).

15 Decidability Problems

Remark 15.1 (Dehn's Problems)**.**

(i) Let G be a group with generating set X. In "practice" one wants to solve the following tasks algorithmically:

- (Word problem) When does a word in X represent the identity element in G?
- (Conjugacy problem) When are two words in X conjugate as elements of G?
- (Isomorphism problem) When is G isomorphic to another given group H?

(ii) A solution to the word problem means that one can bring every element in G into a "normal form". According to Lemma 1.7, the word problem is therefore solvable for free groups (provided one already knows that G is free).

(iii) If the conjugacy problem for G is solvable, then so is the word problem, because $g = 1$ if and only if g and 1 are conjugate.

(iv) Novikov and Boone have shown that all three problems are undecidable even for finitely presented groups, d. h. there is no general algorithm that solves any of the three problems in finite time. Even the question of whether a given group is finite or trivial cannot be decided!

(v) In general, the solvability of the problems depends on the chosen representation. For finitely generated groups, the situation is better.

Theorem 15.2. *Let $G = \langle X \mid R\rangle$ be finitely generated. If the word problem (or conjugacy problem) is solvable with respect to this presentation of G, then the word problem (or conjugacy problem) is also solvable for every other finitely generated presentation of G.*

Proof. Let $F := F_X$ and $\varphi\colon F \to G$ be the canonical epimorphism. Let $F_1 := F_{X_1}$ and $\varphi_1\colon F_1 \to G$ be another presentation with $|X_1| < \infty$. Then there exists a function $\psi\colon X_1 \to F$ with $(\varphi\psi)(x) = \psi_1(x)$ for all $x \in X_1$. By the universal property, ψ extends to F_1. This extension can be explicitly calculated using the finitely many values $\varphi(x)$ with $x \in X_1$. Now let $w \in F_1$. If the word problem for $\langle X \mid R\rangle$ is solvable, then $\varphi(\psi(w)) = 1$ can be decided. Thus, $\varphi_1(w) = 1$ can also be decided, d. h. the word problem for $\langle X_1 \mid R_1\rangle$ is solvable. Analogously for the conjugacy problem. □

Theorem 15.3. *The conjugacy problem is solvable for all free groups.*

Proof. Apparently, every $g \in F_X$ is conjugate to a cyclically reduced word $\breve{g}$. If $\breve{g}$ and $\breve{h}$ differ only by shifts, then g and h are conjugate in F_X. Conversely, assume that g and h are conjugate. Then $\breve{g}$ and $\breve{h}$ are also conjugate. Let $a \in F_X$ be reduced with $a\breve{g}a^{-1} = \breve{h}$. Since $\breve{h}$ is cyclically reduced, a^{-1} must cancel completely with $\breve{g}$. Thus $\breve{g}$ is a shift of $\breve{h}$. In this way, one can decide whether g and h are conjugate. □

Theorem 15.4. *Let G be finitely presented and residually finite. Then the word problem for G is solvable.*

Proof. Let $G = \langle X \mid R \rangle$ be a finite presentation and w a word in X. We execute the following algorithms in parallel (or alternately):

- Construct all (countably many) words in R (for example in lexicographical order). If $w = 1$ in G, then w must eventually appear as such a word.
- Construct all (countably many) finite groups H. Since X is finite, there are only finitely many homomorphisms $\varphi\colon G \to H$. Check $\varphi(w) \neq 1$. If $w \neq 1$, then there exist H and φ with $\varphi(w) \neq 1$, since G is residually finite. □

Theorem 15.5. *The word problem is solvable for Coxeter groups.*

Proof. Let $G = \langle x_1, \ldots, x_n \rangle$ be a Coxeter group of rank n and $w = x_{i_1} \ldots x_{i_l} \in G$. Let $\sigma\colon G \to \mathrm{GL}(n, \mathbb{R})$ be the monomorphism from Theorem 10.14. Then $\sigma(w) = \sigma_{i_1} \ldots \sigma_{i_l}$ can be calculated solely from the numbers m_{ij}. (On a computer, one could realize the matrix entries $\cos(\pi/m_{ij})$ discretely in a cyclotomic field instead of working with rounding-prone floating-point numbers.) It holds that $w = 1$ if and only if $\sigma(w) = 1$. □

Remark 15.6.

(i) Tits has given an efficient algorithm for Theorem 15.5: Let $G = \langle x_1, \ldots, x_n \rangle$ be a Coxeter group and $\pi\colon F_X \to G$ the canonical epimorphism with $X = \{x_1, \ldots, x_n\}$. For $w = x_{i_1} \ldots x_{i_k} \in F_X$ let $R(w) \subseteq F_X$ be the set of all words that can be obtained from w by means of the following operations:

- Replace $x_i x_j x_i \ldots$ (m_{ij} letters) by $x_j x_i x_j \ldots$ (m_{ij} letters).
- If $x_i = x_{i+1}$, then remove $x_i x_{i+1}$.

Since the length of the words in $R(w)$ is bounded, $R(w)$ is finite. Obviously $\pi(r) = \pi(w)$ holds for all $r \in R(w)$. One can show that $\pi(w) = 1$ holds if and only if $1 \in R(w)$ (without proof).

(ii) The isomorphism problem for Coxeter groups has not yet been completely solved.

(iii) The difficulty of Dehn's problems is exploited in cryptography. We describe as an example the Anshel-Anshel-Goldfeld *protocol* for determining a shared secret key between persons A and B. Given a group G for which the word problem is "efficiently" solvable, but the conjugation problem is not. The public key of A and B respectively consists of random elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. The private keys consist of words w_A and w_B in $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. The key exchange is based on the following principle:

(1) A sends $(w_A b_1 w_A^{-1}, \ldots, w_A b_n w_A^{-1})$ to B.

(2) B sends $(w_B a_1 w_B^{-1}, \ldots, w_B a_n w_B^{-1})$ to A.

(3) Both can now calculate

$$w_A w_A(w_B a_1 w_B^{-1}, \ldots, w_B a_n w_B^{-1}) = [w_A, w_B] = w_B(w_A b_1 w_A^{-1}, \ldots, w_A b_n w_A^{-1}) w_B^{-1}$$

and use it as a shared key. In practice, braid groups or polycyclic groups are used for G. These cryptographic methods become interesting when previous methods like RSA become unusable with powerful quantum computers.

Exercises

Exercise 1. Let F be a free group of rank > 1. Show $\mathrm{Z}(F) = 1$.

Exercise 2. Show:

(a) $Q_{2^n} \cong \langle x, y \mid x^{2^{n-2}} = y^2,\ yxy^{-1} = x^{-1} \rangle$ for $n \geq 3$ (see GT-Theorem 8.15)

(b) $A_4 \cong \langle x, y \mid x^2 = y^3 = (xy)^3 = 1 \rangle$.

Exercise 3. A group G is called *metacyclic*, if a cyclic normal subgroup N with cyclic factor group G/N exists (Example: GT-Theorem 7.24). Let P be a finite metacyclic p-group. Show that $a, b, c, k \geq 0$ with $k^{p^b} - 1 \equiv 0 \equiv p^c(k-1) \pmod{p^a}$ and

$$P \cong \langle x, y \mid x^{p^a} = 1,\ y^{p^b} = x^{p^c},\ yxy^{-1} = x^k \rangle$$

exist.
Remark: Different parameters can belong to isomorphic groups. An exact classification was given by Liedahl.

Exercise 4. Let p be a prime, $a \geq 2$ and $b \geq 1$. Let

$$P(a,b) := P = \langle x, y \mid x^{p^a} = y^{p^b} = 1,\ yxy^{-1} = x^{1+p^{a-1}} \rangle.$$

Show:

(a) $|P| = p^{a+b}$.

(b) $P' = \langle x^{p^{a-1}} \rangle \cong C_p$.

(c) $\Phi(P) = \mathrm{Z}(P) = \langle x^p, y^p \rangle \cong C_{p^{a-1}} \times C_{p^{b-1}}$.

Exercise 5. A non-abelian group G is called *minimal non-abelian*, if every proper subgroup of G is abelian. Show that for a finite p-group P the following statements are equivalent:

(a) P is minimal non-abelian.

(b) $|P : \Phi(P)| = |P : \mathrm{Z}(P)| = p^2$.

(c) $|P : \Phi(P)| = p^2$ and $|P'| = p$.

Hint: GT-Chapter 4.

Exercise 6. Let p be a prime and $a, b \in \mathbb{N}$. Let

$$Q(a,b) := Q = \langle x, y \mid x^{p^a} = y^{p^b} = [x,y]^p = [x,x,y] = [y,x,y] = 1 \rangle$$

(Reminder: $[x,y,z] := [x,[y,z]]$). Show:

(a) $|Q| = p^{a+b+1}$.

(b) $Q' = \langle [x,y] \rangle \cong C_p$.

(c) $\Phi(Q) = \mathrm{Z}(Q) = \langle x^p, y^p, [x,y] \rangle \cong C_{p^{a-1}} \times C_{p^{b-1}} \times C_p$.

Remark: Rédei has shown that every minimal non-abelian p-group is isomorphic to $P(a,b)$, $Q(a,b)$ or to Q_8.

Exercise 7. Decide whether the Petersen graph is the Cayley graph of a group.

Exercise 8. Let F be the free group over the alphabet $\{x,y\}$ and $G \leq F$ the subgroup of all words with even length. Show that G is freely generated by x^2, y^2 and xy.

Exercise 9. Let $G = \langle X \mid R \rangle$ be a simple group, where every relator in R has even length. Show: $G \cong C_2$.

Exercise 10. Let F be a free group and $w \in F \setminus \{1\}$. Show $\mathrm{C}_F(w) \cong C_\infty$. Conclude that F is indecomposable.

Exercise 11. Show that a group H is free if and only if every extension of H splits.

Exercise 12. The *coproduct*

$$A := \coprod_{n \in \mathbb{Z}} \mathbb{F}_2 \leq \prod_{n \in \mathbb{Z}} \mathbb{F}_2$$

consists of all sequences $(a_n)_{n \in \mathbb{Z}}$ with $|\{n \in \mathbb{Z} : a_n = 1\}| < \infty$. Obviously, there exists $\gamma \in \mathrm{Aut}(A)$ with $\gamma((a_n)_n) := (a_{n+1})_n$ for all $n \in \mathbb{Z}$. Let $G := A \rtimes \langle \gamma \rangle$. Show that G is finitely generated and metabelian, but the subgroup $A \leq G$ is not finitely generated.

Exercise 13. Determine all extensions of C_2 by C_n up to equivalence ($n \in \mathbb{N}$).

Exercise 14. Let

$$D_\infty := \langle x, y \mid x^2 = y^2 = 1 \rangle \cong C_\infty \rtimes C_2$$

(GT-Exercise 61(b)). Show that $G := \langle x, y \mid x^2 = y^2 \rangle$ is an extension of D_∞ by $\mathrm{Z}(G) \cong C_\infty$. Conclude that G is supersolvable and G' is cyclic.

Exercise 15. Let $A := \langle a \rangle \cong C_4$, $S := \mathrm{SL}(2,3)$ and $G := (S \times A)/\langle (-1_2, a^2) \rangle \cong S * A$ (central product). Show:

(a) $S \cong Q_8 \rtimes C_3$.

(b) $Q_8 \trianglelefteq G$ possesses a complement in $H := Q_8 * A \in \mathrm{Syl}_2(G)$.

(c) Q_8 possesses no complement in G.

Remark: For non-abelian groups N, Theorem 4.23 by Gaschütz is therefore false.

Exercise 16. Let $N := D_8$ and $H \cong C_2$.

(a) Show $\mathrm{Aut}(N) \cong D_8$.

(b) Determine all extensions of H by N up to equivalence. Which of these are isomorphic?

Exercise 17. Let $N = \langle a, b \mid a^8 = b^2 = 1,\ bab^{-1} = a^{-1} \rangle \cong D_{16}$ and $H = \langle x \rangle \cong C_2$. Show:

(a) There exists an automorphism $\beta \in \mathrm{Aut}(N)$ with $\beta(a) = a^3$ and $\beta(b) = ab$.

(b) There exists a homomorphism $\omega\colon H \to \mathrm{Out}(N)$ with $\omega(x) = \beta\mathrm{Inn}(N)$.

(c) There is no system of parameters of H by N for the coupling ω.

Hint: Theorem 4.16.

Exercise 18. We consider the simple group $N := \mathrm{PSL}(2, 9)$ of order 360. We identify the elements in N with their preimages in $\mathrm{SL}(2, 9)$ (note: $\mathrm{Z}(\mathrm{SL}(2, 9)) = \langle -1_2 \rangle$). Let $\mathbb{F}_9^\times = \langle \zeta \rangle$ and $d := \mathrm{diag}(\zeta, 1) \in \mathrm{GL}(2, 9)$. Show:

(a) The map $\sigma\colon N \to N$, $x \mapsto dxd^{-1}$ is an outer automorphism with $\sigma^2 \in \mathrm{Inn}(N)$.

(b) The map $\tau\colon N \to N$, $(x_{ij}) \mapsto (x_{ij}^3)$ is an outer automorphism of order 2.

(c) Let $H = \langle x \rangle \cong C_2$. According to Corollay 4.28, there is one system of parameters (α, κ) of H by N for each $\alpha_x \in \{1, \sigma, \tau, \sigma\tau\}$. Investigate which of the extensions split and which are isomorphic.

Remark: It holds that $N \cong A_6$.

Exercise 19. Let N be abelian and $\alpha\colon H \to \mathrm{Aut}(N)$ a group homomorphism. A map $\delta\colon H \to N$ with

$$\boxed{\delta(xy) = \delta(x)\alpha_x(\delta(y))}$$

for all $x, y \in H$ is called a *crossed homomorphism* w.r.t. α. Show that:

(a) The crossed homomorphisms form a group $\mathrm{Hom}_\alpha(H, N) \leq C^1(H, N)$.

(b) The map $\Gamma\colon N \to \mathrm{Hom}_\alpha(H, N)$, $a \mapsto \delta_a$ with $\delta_a(x) := \alpha_x(a)a^{-1}$ is a homomorphism. One sets $H^1_\alpha(H, N) := \mathrm{Hom}_\alpha(H, N)/\Gamma(N)$.

(c) Let K be a complement of N in $G := N \rtimes_\alpha H$. For $x \in H$ there exists exactly one $y \in K$ with $\delta_K(x) := xy^{-1} \in N$. It holds that $\delta_K \in \mathrm{Hom}_\alpha(H, N)$.

(d) Every $\delta \in \mathrm{Hom}_\alpha(H, N)$ defines a complement $K_\delta := \{\delta(x)^{-1}x : x \in H\}$ of N in G with $\delta_{K_\delta} = \delta$.

(e) Two complements K_1, K_2 of N in G are conjugate if and only if $\delta_{k_1}\delta_{K_2}^{-1} \in \Gamma(N)$ holds.

(f) The map $K \to \delta_K$ induces a bijection between the conjugacy classes of complements of N in G and $H^1_\alpha(H, N)$.

Exercise 20 (Gaschütz)**.** Let N be an abelian normal subgroup of a finite group G and $N \leq H \leq G$ with $\gcd(|N|, |G : H|) = 1$. Show that: If all complements of N in H are conjugate, then all complements of N in G are conjugate.

Exercise 21.

(a) Show that $\mathrm{Aut}(S_n) \cong \mathrm{Aut}(A_n)$ for $n \geq 4$.

(b) Show that $\varphi \in \mathrm{Aut}(S_6)$ with

$$\begin{aligned}
&\varphi((1,2)) = (1,5)(2,3)(4,6), && \varphi((1,3)) = (1,4)(2,6)(3,5),\\
&\varphi((1,4)) = (1,3)(2,4)(5,6), && \varphi((1,5)) = (1,2)(3,6)(4,5),\\
&\varphi((1,6)) = (1,6)(2,5)(3,4)
\end{aligned}$$

is an outer automorphism of order 2.

Exercise 22. Let $P = P(a,1) = M_{p^{a+1}}$ be the minimal non-abelian group from Exercise 4 (or GT-Theorem 8.15) with $a \geq 2$. In the case $p = 2$ let $a \geq 3$. Show that $M(P) = 1$.
Hint: Theorem 5.25.

Exercise 23. Let $G_i \approx H_i$ be isoclinic groups for $i = 1, 2$. Show that $G_1 \times G_2 \approx H_1 \times H_2$.

Exercise 24. Show that one must fill out at least 82 lottery tickets (6 out of 49) to have two "correct numbers" (on one ticket).

Exercise 25. Let $S = (\Omega, \mathcal{B})$ be a $(2,k,v)$-Steiner system. Show that the following statements are equivalent:

(1) $v = k^2$.

(2) $|\mathcal{B}| = k^2 + k$.

(3) For $B \in \mathcal{B}$ and $\omega \in \Omega \setminus B$ there exists exactly one block $B' \in \mathcal{B}$ with $\omega \in B'$ and $B \cap B' = \varnothing$.

Remark: In the affine plane $\mathbb{F}_q^2$, (3) is the *parallel postulate*: For every point x and every line g there exists exactly one parallel of g passing through x.

Exercise 26. Show that the Higman-Sims graph is 22-regular, i. e. every vertex is connected to exactly 22 other vertices.

Exercise 27.

(a) Show that $\mathrm{SU}(3,3)$ has exactly 63 involutions.

(b) Show with GAP that the graph Γ defined for J_2 in Remark 9.20(viii) is 36-regular.

(c) Show $\mathrm{Aut}(\Gamma) \cong J_2 \rtimes C_2$ using the packages `grape` and `atlasrep`.

Exercise 28. A *lattice* is a free abelian subgroup of $\mathbb{R}^n$ of rank n. Every lattice thus has the form $L = \mathbb{Z}b_1 + \ldots + \mathbb{Z}b_n$, where $b_1, \ldots, b_n \in \mathbb{R}^n$ is a basis of $\mathbb{R}^n$. Show that

$$\mathrm{Aut}(L) = \{f \in \mathrm{O}(\mathbb{R}^n) : f(L) = L\}$$

is a finite group. Determine $\mathrm{Aut}(\mathbb{Z}^n)$.

Exercise 29. Show that an action of G on Ω is 2-transitive if and only if $G = G_\omega \cup G_\omega x G_\omega$ for $\omega \in \Omega$ and an $x \in G$ holds.

Exercise 30. Show that the simple group $\mathrm{PSp}(4,3)$ is not isomorphic to any alternating group and any projective special linear group.
Remark: According to Example 8.13, $\mathrm{PSp}(4,3) \cong \mathrm{PSU}(4,2)$ holds. With the help of *Zsigmondy primes* one can show that $\mathrm{PSp}(2n,q)$ for $n \geq 2$ is not isomorphic to any alternating group and any projective special linear group.

Exercise 31. Let V be a unitary space. Show that V has a basis consisting of elements in V_0.

Exercise 32. Let V be a unitary space of dimension n and $U \leq V$ with $U \subseteq U^\perp$. Show $\dim U \leq \lfloor n/2 \rfloor$. Give an example in which equality holds.
Remark: In general (i. e. for arbitrary "scalar products" on V) one calls

$$\max\{\dim U : U \leq V,\ U \subseteq U^\perp\}$$

the *Witt index* of V.

Exercise 33. Show $\mathrm{Sp}(2n,q) \leq \mathrm{SU}(2n,q)$ for all $n \in \mathbb{N}$ and prime powers $q \neq 1$.

Exercise 34. Prove $\mathrm{PSU}(3,2) \cong M_9$.
Hint: GT-Theorem 6.21.

Exercise 35. Let $q \neq 1$ be an odd prime power and V be an n-dimensional $\mathbb{F}_q$-vector space. Let $\beta\colon V \times V \to \mathbb{F}_q$ be a non-degenerate symmetric bilinear form. Let $\lambda \in K^\times \setminus (K^\times)^2$ be a non-square.

(a) Construct a basis $b_1, \ldots, b_n$ of V with $\beta(b_1,b_1) \in \{1,\lambda\}$, $\beta(b_i,b_i) = 1$ for $i = 2,\ldots,n$ and $\beta(b_i,b_j) = 0$ for $i \neq j$.

(b) Let β_1 and β_2 be bilinear forms representing the two possibilities in (a). Let n be odd. Show $\mathrm{GO}(n,q,\beta_1) \cong \mathrm{GO}(n,q,\beta_2)$ with the notation from Remark 8.28.

(c) Let $n = 2$. Show that for exactly one of the two bilinear forms there exists a $v \in V \setminus \{0\}$ with $\beta(v,v) = 0$. We denote this form by β_+ (Witt index 1) and the other by β_- (Witt index 0). For $\epsilon = \pm 1$ let $\mathrm{GO}^\epsilon(2,q) := \mathrm{GO}(2,q,\beta_\epsilon)$. Show $\mathrm{GO}^\epsilon(2,q) \cong D_{2(q-\epsilon)}$.

Remark: According to Sylvester's law of inertia, there are exactly $n+1$ non-equivalent non-degenerate symmetric bilinear forms on $\mathbb{R}^n$.

Exercise 36. Let $p > 2$ be a prime and $G := \langle x,y \mid x^p = y^p = (xy)^p = 1\rangle$. Show $|G| = \infty$.
Hint: Realize G as a subgroup of $\mathrm{Sym}(\mathbb{Z})$.

Exercise 37. Show:

(a) Every Hurwitz group G is perfect and 84 divides $|G|$.

(b) $\mathrm{GL}(3,2)$ is a Hurwitz group.

Exercise 38. Let $G = \mathrm{Fr}_{i\in I} G_i$ and $H = \bigoplus_{i\in I} G_i$.

(a) Construct a natural epimorphism $G \to H$ and describe its kernel.

(b) Show $G/G' \cong H/H'$.

(c) Let $N_i \trianglelefteq G_i$ for $i \in I$. Let N be the normal closure of $\bigcup_{i\in I} N_i$ in G. Show $G/N \cong \mathrm{Fr}_{i\in I}\, G_i/N_i$.

Exercise 39. Let G be an amalgam of G_I with respect to H with $H < G_i$ for all $i \in I$. Show: $\mathrm{Z}(G) = \bigcap_{i\in I} \mathrm{Z}(G_i)$.

Exercise 40. Show that every finitely generated periodic solvable group is finite.

Exercise 41. Write the element c_3 in the Hall-Petrescu formula in a form that shows it lies in $G^{[3]}$.

Exercise 42.

(a) Let P be a p-group such that $P^{[p-1]}$ is cyclic. Show that P satisfies the statements of Lemma 14.15.

(b) Construct p-groups with $p > 2$ as in (a) with arbitrarily large nilpotency class.

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